Inverse boundary value problems with partial and local data for the magnetic Schrödinger operator

Leyter Potenciano Machado

Universidad Autónoma de Madrid

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Supervised by Alberto Ruiz González

A mi familia.

Ítaca

Cuando emprendas tu viaje hacia Ítaca
debes rogar que el viaje sea largo,
lleno de peripecias, lleno de experiencias.
No has de temer ni a los lestrigiones ni a los cíclopes,
ni la cólera del airado Posidón.
Nunca tales monstruos hallarás en tu ruta
si tu pensamiento es elevado, si una exquisita
emoción penetra en tu alma y en tu cuerpo.
Los lestrigones y los cíclopes
y el feroz Posidón no podrían encontrarte
si tú no los llevas ya dentro, en tu alma,
si tu alma no los conjura ante ti.
Debes rogar que el viaje sea largo,
que sean muchos los días de verano;
que te vean arriba con gozo, alegremente,
a puertos que tú antes ignorabas.
Que puedas detenerte en los mercados de Fenicia,
y comprar unas bellas mercancías:
madreperlas, coral, ébano, y ámbar,
y perfumes placenteros de mil clases.
Acude a muchas ciudades de Egipto
para aprender, y aprender de quienes saben.
Conserva siempre en tu alma la idea de Ítaca:
llegar allí, he aquí tu destino.
Mas no hagas con prisa tu camino;
mejor será que dure muchos años,
y que llegues, ya viejo, a la pequeña isla,
rico de cuanto habrías ganado en el camino.
No has de esperar que Ítaca te enriquezca:
Ítaca te ha concedido ya un hermoso viaje.
Sin ellas, jamás habrías partido;
mas no tiene otra cosa que ofrecerte.
Y si lo encuentras pobre, Ítaca no te ha engañado.
Y siendo ya tan viejo, con tanta experiencia,

sin duda sabrás ya qué significan las Ítacas.

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Resumen

El objetivo principal de esta disertación consiste en estudiar dos problemas inversos asociados a un operador de Schrödinger con término magnético, en un dominio acotado. Presentamos unos resultados sobre la recuperación y la recuperación estable de los coeficientes de dicho operador a partir de mediciones en la frontera del dominio. Físicamente hablando, estos coeficientes representan los potenciales magnético y eléctrico del operador.

Abordaremos dos problemas, los cuales dependen del tipo de mediciones que haremos en la frontera. Ambos problemas tienen en común que las mediciones se toman en subconjuntos abiertos de la frontera. Para el primer problema estudiado, tomamos las mediciones en un subconjunto por medio de perturbaciones en el complemento del mismo. En este caso, aunque las mediciones se hacen en subconjuntos, tenemos acceso al complemento de los mismos para poder hacer las perturbaciones. Por el contrario, en el segundo problema hay una parte inaccesible, por lo cual las perturbaciones y las correspondientes mediciones se hacen en el mismo conjunto, llamado la parte accesible de la frontera.

En ambos casos, recuperamos el campo magnético y el potencial eléctrico del operador. Además, derivamos las correspondientes estimaciones de estabilidad, obteniendo módulos de continuidad logarítmicos. Respecto a las estimaciones para el primer problema, obtenemos un doble logaritmo como módulo de continuidad para el campo magnético y un triple para el potencial eléctrico. Para enfrentar el segundo problema, debido a que solo tenemos acceso a una parte de la frontera, imponemos una restricción geométrica: la parte inaccesible se encuentra contenida en un hiperplano. Bajo esta hipótesis adicional, obtenemos estimaciones con un solo logaritmo como módulo de continuidad, tanto para el campo magnético como para el potencial elétrico.

Finalmente mencionamos que, por lo general, el estudio de la recuperación estable con mediciones en parte de la frontera implica, por lo menos, un doble logaritmo como módulo de continuidad. En ese sentido, el módulo de continuidad logarítmico obtenido en el segundo problema, es el óptimo.

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Abstract

The main goal of this dissertation is the study of two Inverse Boundary Value Problems associated with a magnetic Schrödinger operator on a bounded domain. We give results about the determination and the stable determination of the coefficients of the operators. Physically speaking, such coefficients represent the magnetic and the electric potentials of the operators.

We study two problems, both depending on the kind of measurements taken on the boundary and having in common that the measurements should be taken on open subsets. For the first problem, we take the measurements on a subset of the boundary by means of perturbations on its complement. Thus we say that in this case we may *access* this complement. On the contrary, in the second problem there is an *inaccessible part*, hence the perturbations and the corresponding measurements are taken on the same set, called the *accessible part* of the boundary.

In both problems, we can recover the magnetic field and the electric potential of the operator. Moreover, we derive the corresponding stability estimates, obtaining module of continuity of logarithmic type. Regarding the estimates for the first problem studied, we obtain a double logarithm as a module of continuity for the magnetic field and a triple for the electric potential. To deal with the second problem, since we only have access to a part of the boundary, we impose a geometric restriction: the inaccessible part of the boundary is contained in a hyperplane. Under this additional hypothesis, we obtain estimates with only one logarithm as modulus of continuity for both the magnetic field and the electric potential.

In general, the study of stable determination with measurements on part of the boundary implies twice the logarithm as module of continuity. In this sense, the logarithmic module of continuity obtained in the second problem is optimal.

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Chapter 1

Introduction

An inverse boundary value problem, IBVP for short, consists in the determination of unknown parameters inside a body, from external measurements on its boundary. Most IBVPs arise from physical phenomena and so they are modeled by a Partial Differential Equation. For an elliptic equation, the most relevant example of IBVP is the nowadays known as Calderón's problem. It was formulated by Calderón [6] in his pioneer work entitled "On an inverse boundary value problem". Calderón's motivation was the oil exploration by electrical methods while he was working as an engineer for the Argentinian state oil company. Physically, the problem consists of determining the electrical conductivity of a body by making current and voltage measurements on its boundary. More recently, this kind of problems has become even more important due to wider of applications not only in oil exploration but also in other areas, for example in medical imaging, where the problem is known as Electrical Impedance Tomography (EIT). Further proposed EIT applications are the early diagnosis of breast cancer, the detection/localization of pulmonary edema and imaging of brain activity, among other applications. See for instance [20]-[21] and [31].

The mathematical formulation of the Calderón problem is as follows. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$ and also let $\sigma \in L^\infty(\Omega)$ be a strictly positive function which represents the electric conductivity inside Ω. In this setting and in the absence of sources or sinks of the current, for a given voltage on the boundary f, the voltage potential u inside Ω solves the following conductivity equation:

$$\begin{cases} \operatorname{div}(\sigma\nabla u) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = f & \text{on } \partial\Omega. \end{cases} \tag{1.0.1}$$

The Lax-Milgram theorem ensures the well-posedness of this equation. In particular, for every $f \in H^{1/2}(\partial\Omega)$ there exists a unique $u \in H^1(\Omega)$ solving the conductivity equation (1.0.1). This allows us to define the Dirichlet-Neumann map, DN map for short, $\Lambda_\sigma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$, associated to (1.0.1) by:

$$\Lambda_\sigma f = (\sigma\nu \cdot \nabla u)\,|_{\partial\Omega}, \tag{1.0.2}$$

where ν denotes the exterior unit normal of $\partial\Omega$. This map is linear and bounded. Physically, it encodes the information of the measurements on the boundary of the outgoing current flux $(\sigma\nu\cdot\nabla u)|_{\partial\Omega}$, for a prescribed voltage f on the boundary. In this mathematical framework, the **Calderón problem** consists of determining the electric conductivity σ from the knowledge of the DN map Λ_σ. Associated with this problem, a couple of natural questions arise: the **identifiability** and the corresponding **stability estimates** when determining the electric conductivity.

Q1. Identifiability. The identifiability issue concerns the injectivity of the DN map Λ_σ. That is, given two pairs of electric conductivities σ_1, σ_2 with $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$, does it follow that $\sigma_1 = \sigma_2$ in Ω?

Q2. Stability. The issue of stability concerns a quantitative estimate of the previous qualitative question. That is, ask for the existence of a modulus of continuity m such $m(0) = 0$ and the following estimate:

$$\|\sigma_1 - \sigma_2\|_{L^\infty(\Omega)} \leqslant m\left(\|\Lambda_{\sigma_1} - \Lambda_{\sigma_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}\right),$$

holds true for any two pairs of conductivities σ_1, σ_2, sufficiently close.

Analogous questions were extended to others IBVPs, each one of them coming from different Partial Differential Equations and so with different applications from the practical point of view. See for instance [37]-[38] for the elasticity equation and [43] for the Dirac equation [43]. Now, following Sylvester and Uhlamnn's method [47], we explain briefly how they faced the identifiability question. By setting $u = \sigma^{-1/2}v$, it is immediate to see that

$$\text{div}\,(\sigma\nabla u) = -\sigma^{-1/2}\left(-\Delta + \sigma^{-1/2}\Delta\sigma^{1/2}\right)v. \tag{1.0.3}$$

From this identity, we deduce that given $f \in H^{1/2}(\partial\Omega)$, a function $u \in H^1(\Omega)$ is a solution of the conductivity equation (1.0.1) if and only if $v \in H^1(\Omega)$ is a solution of the following Dirichlet problem:

$$\begin{cases} (-\Delta + \sigma^{-1/2}\Delta\sigma^{1/2})v = 0 & \text{in } \Omega, \\ v|_{\partial\Omega} = \sigma^{1/2}f & \text{on } \partial\Omega. \end{cases} \tag{1.0.4}$$

Motivated from this relation, they considered the following Schrödinger equation:

$$\begin{cases} (-\Delta + q)v = 0 & \text{in } \Omega, \\ v|_{\partial\Omega} = g & \text{on } \partial\Omega. \end{cases} \tag{1.0.5}$$

Usually, the function q is called the electric potential. Next, they defined a new IBVP associated to (1.0.5) as follows. By assuming that $q \in L^\infty(\Omega)$ and that 0 is not an eigenvalue in $L^2(\Omega)$ of the Laplacian operator Δ, the Fredholm Alternative theorem ensures that for every $g \in H^{1/2}(\partial\Omega)$ there exists a unique $v \in H^1(\Omega)$ satisfying (1.0.5). This allowed them to define a new DN map $\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ as

$$\Lambda_q(g) = (\nu\cdot\nabla u)|_{\partial\Omega}. \tag{1.0.6}$$

Thus, the new IBVP associated to (1.0.5) consists of determining the electric potential q from the knowledge of the DN map Λ_q. Later, they proved that the identifiability of the IBVP associated to the Schrödinger equation (1.0.5) implies the identifiability for the Calderon problem, i.e. if $\Lambda_{q_1} = \Lambda_{q_2}$ implies $q_1 = q_2$ then $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$ implies $\sigma_1 = \sigma_2$, this requires the recovery of the values of σ and $\partial_\nu \sigma$ on the boundary. Hence, they gave a positive answer to the identifiability question for the Calderon problem by proving first the identifiability for the IBVP associated to (1.0.5).

Due to this connection, almost all efforts were focused on studying fully the IBVP associated to the Schrödinger equation (1.0.5). See for instance [51] and also [52] for a complete survey of these problems. Now we move on to describing an IBVP which is a natural generalization of the previous IBVP for the Schrödinger equation. It will be our starting point to introduce our problems and results. A natural extension of the operator $-\Delta + q$ is obtained by introducing into it a magnetic potential, denoted by A. In this case, the new operator is usually called the magnetic Schrödinger operator. More precisely, we consider the magnetic Schrödinger operator $\mathcal{L}_{A,q}$, defined by

$$\mathcal{L}_{A,q}(x, D) := \sum_{j=1}^{n} (D_j + A_j(x))^2 + q(x) = -\Delta + A \cdot D + D \cdot A + A^2 + q, \tag{1.0.7}$$

with $D = -i\nabla$, the vectorial function A represents a magnetic potential, the scalar function q represents an electric potential and $A^2 = \sum_{j=1}^{n} A_j^2$. Notice that if $A \equiv 0$ then $\mathcal{L}_{0,q} = -\Delta + q$. Now we consider the initial BVP

$$\begin{cases} \mathcal{L}_{A,q}\, u = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = f & \text{on } \partial\Omega. \end{cases} \tag{1.0.8}$$

By assuming that 0 is not an eigenvalue in $L^2(\Omega)$ of the magnetic Schrödinger operator $\mathcal{L}_{A,q} : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$, the Fredholm Alternative Theorem ensures that for every $f \in H^{1/2}(\partial\Omega)$ there exists a unique solution $u \in H^1(\Omega)$ of (1.0.8). This allows us to define the global DN map associated to the magnetic Schrödinger equation (1.0.8) as:

$$\begin{aligned} \Lambda_{A,q} :& H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega) \\ & f \to (\partial_\nu + iA \cdot \nu) u|_{\partial\Omega}, \end{aligned} \tag{1.0.9}$$

where ν is the exterior unit normal of $\partial\Omega$. Notice that if $A \equiv 0$, we have the DN map $\Lambda_q = \Lambda_{0,q}$ defined in (1.0.6). Notice also that in the equation (1.0.8) we now have two parameters, namely A and q. Consequently, the global IBVP problem associated to the magnetic Schrödinger equation (1.0.8) consists of determining both the magnetic potential A and the electric potential q from the knowledge of the DN map $\Lambda_{A,q}$. This problem is also called IBVP for the magnetic Schrödinger operator. Analogously to the previous IBVPs, this problem was also intensively studied by several authors, mainly focused on the issues of identifiability and the corresponding stability estimates. Throughout this dissertation, we consider the magnetical potential A as a 1-form as follows

$$A = \sum_{j=1}^{n} A_j dx_j, \quad A = (A_1, A_2, \ldots, A_n)$$

and

$$dA = \sum_{1 \leqslant j < k \leqslant n} \left(\partial_{x_j} A_k - \partial_{x_k} A_j\right) d_{x_j} \wedge d_{x_k}.$$

As it was noted in [46], in the presence of a magnetic potential ($A \not\equiv 0$) there exists a gauge invariance of the DN map $\Lambda_{A,q}$. To be specific, if $\varphi \in C^1(\overline{\Omega})$ is a real-valued function with $\varphi|_{\partial\Omega} = 0$, then $\Lambda_{A,q} = \Lambda_{A+\nabla\varphi,q}$. Hence, for the identifiability problem, we only expect to prove that $dA_1 = dA_2$, in physic this is known as the magnetic field, and $q_1 = q_2$ in Ω.

Q3. Identifiability. Given two magnetic potentials A_1, A_2 and two electric potentials q_1, q_2 with $\Lambda_{A_1,q_1} = \Lambda_{A_2,q_2}$, does it follow that $dA_1 = dA_2$ and $q_1 = q_2$ in Ω?

Q4. Stability. Does there exist a modulus of continuity m with $m(0) = 0$ such that the following estimate:

$$\|dA_1 - dA_2\|_{L^2(\Omega)} + \|q_1 - q_2\|_{L^2(\Omega)} \leqslant m\left(\|\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}\right),$$

holds true for all magnetic potentials A_1, A_2 and for all electric potentials q_1, q_2, sufficiently close?

The first results on identifiability and stability estimates were obtained when the knowledge of the DN map is on the whole boundary. These cases are usually called full data cases. But, from the practical point of view, it might be too difficult or, even worse, impossible to measure on all the boundary. For instance, one could think of the oil exploration in the ocean. We can only cover a small part of the ocean's surface with measurement devices. Hence, due to the practical applications, several authors dedicated many efforts to the study of the IBVP associated to the Schrödinger equation (1.0.5) and, more recently, to the IBVP associated to the magnetic Schrödinger equation (1.0.8), both with partial and local data . That is, in the case when the knowledge of the DN map is only assumed on some subset of the boundary for functions supported in the same (local data) or another (partial data) subset of the boundary.

In this dissertation, we study two IBVPs, both associated to the magnetic Schrödinger operator $\mathcal{L}_{A,q}$. The first problem is relative to a partial data case. The second is relative to a local data case. We prove, in both cases, that measuring on some part of the the boundary is sufficient to determine the magnetic field and the electric potential of the operator $\mathcal{L}_{A,q}$. We also derive the corresponding stability estimates.

Throughout this dissertation we denote by C_i, $i \in \mathbb{Z}^+$, a positive constants which might change from formula to formula. These constants should depend only on n, Ω and the priori bounds for magnetic and electrical potentials. Also, for given $E \subset \mathbb{R}^n$ a bounded open set and any function $h : E \to \mathbb{C}$ (or $\mathbb{C}^n$) or $h : E \to \mathbb{R}$ (or $\mathbb{R}^n$) , we denote by $\chi_E\, h$ the extension by zero of h out of E.

1.1 Our first set of results. Illuminating Ω from the infinity

In this section we state our first results. We consider an IBVP with partial data associated to the magnetic Schrödinger equation (1.0.8). Roughly speaking, we consider the measurements on subsets of the boundary for functions supported on the complement of these subsets. Before stating our results, we introduce some assumptions, notations and definitions.

- ◇ **Assumption** 1. Assume that $\Omega \subset \mathbb{R}^n$ is a simply connected bounded open set with smooth boundary $\partial\Omega$.
- ◇ **Assumption** 2. Assume that the magnetic potential A belongs to $C^{2,\gamma}\left(\overline{\Omega};\mathbb{R}^n\right)$ with $\gamma \in (0,1)$, and the electric potential q belongs to $L^\infty(\Omega;\mathbb{R})$. Here $C^{2,\gamma}\left(\overline{\Omega};\mathbb{R}^n\right)$ denoted the space consisting of all functions in $C^2\left(\overline{\Omega};\mathbb{R}^n\right)$ such that its second derivatives belong to the Hölder space $C^\gamma\left(\overline{\Omega};\mathbb{R}^n\right)$.
- ◇ **Assumption** 3. Assume that 0 is not an eigenvalue in $L^2(\Omega)$ of the magnetic Schrödinger operator $\mathcal{L}_{A,q} : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$.

Now we introduce the partial DN map and the IBVP with partial data for the magnetic Schrödinger operator when illuminating Ω from the infinity. Given a direction $\xi \in S^{n-1}$ and $\varepsilon \geqslant 0$, we define the (ξ,ε)-illuminated face of $\partial\Omega$ as

$$\partial\Omega_{-,\varepsilon}(\xi) = \{x \in \partial\Omega \ : \ \langle \xi, \nu(x)\rangle < \varepsilon\}, \tag{1.1.1}$$

and the (ξ,ε)-shadowed face as

$$\partial\Omega_{+,\varepsilon}(\xi) = \{x \in \partial\Omega \ : \ \langle \xi, \nu(x)\rangle > -\varepsilon\},$$

where $\nu(x)$ denotes the exterior unit normal vector at x. Let N be an open subset of S^{n-1} and define the sets

$$F_N = \bigcup_{\xi\in N} \partial\Omega_{-,0}(\xi), \quad B_N = \bigcup_{\xi\in N} \partial\Omega_{+,0}(\xi). \tag{1.1.2}$$

Now let F and B be open neighborhoods on $\partial\Omega$ of F_N and B_N, respectively; and let χ be any cutoff function supported on F such that it equals 1 on F_N. Denote by $H_B^{1/2}(\partial\Omega)$ the set consisting of all the functions $f \in H^{1/2}(\partial\Omega)$ such that $\operatorname{supp} f \subset \overline{B}$. In this framework, we define the partial DN map, $\Lambda_{A,q}^\sharp : H_B^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$, as follows:

$$\Lambda_{A,q}^\sharp f = \chi\Lambda_{A,q} f. \tag{1.1.3}$$

We also consider the associated operator norm defined by

$$\left\|\Lambda_{A,q}^\sharp\right\|_{H_B^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} = \sup_{\substack{f\in H_B^{1/2}(\partial\Omega)\\ \|f\|_{H^{1/2}(\partial\Omega)}=1}} \|\chi\Lambda_{A,q}f\|_{H^{-1/2}(\partial\Omega)}. \tag{1.1.4}$$

The IBVP with partial data under consideration in this section consists of determining the magnetic field dA and the electric potential q from the knowledge of the partial DN map $\Lambda^\sharp_{A,q}$. From the gauge invariance of the global DN map $\Lambda_{A,q}$ defined in (1.0.9), and since the cutoff function is supported in F and it equals 1 on F_N it follows that the partial DN map $\Lambda^\sharp_{A,q}$ has the same kind of gauge invariance, that is, the equality $\Lambda^\sharp_{A,q} = \Lambda^\sharp_{A+\nabla\varphi,q}$ holds true for every real function $\varphi \in C^1(\overline{\Omega})$ with $\varphi|_{\partial\Omega} = 0$. Hence, for the identifiability problem of the partial data case considered in this dissertation, as in the the full data case, we only expect to prove that $dA_1 = dA_2$ and $q_1 = q_2$ in Ω. More precisely, we give answers to the following questions:

⊛ **Identifiability**. Given two magnetic potentials A_1, A_2 and two electric potentials q_1, q_2 such that $\Lambda^\sharp_{A_1,q_1} = \Lambda^\sharp_{A_2,q_2}$, does it follow that $dA_1 = dA_2$ and $q_1 = q_2$ in Ω?

⊛ **Stability**. Does there exist a modulus of continuity m such that the following estimate:

$$\|dA_1 - dA_2\|_{L^2(\Omega)} + \|q_1 - q_2\|_{L^2(\Omega)} \leqslant m\left(\left\|\Lambda^\sharp_{A_1,q_1} - \Lambda^\sharp_{A_2,q_2}\right\|_{H_B^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}\right),$$

holds true for all magnetic potentials A_1, A_2 and for all electric potentials q_1, q_2?

Our identifiability result is stated as follows.

Theorem 1.1.1. *Let $\gamma \in (0,1)$. Let $A_1 \in C^{2,\gamma}(\overline{\Omega};\mathbb{R}^n)$ and $A_2 \in C^2(\overline{\Omega};\mathbb{R}^n)$ be two real magnetic potentials with $A_1 = A_2$ on $\partial\Omega$ and let $q_1, q_2 \in L^\infty(\Omega;\mathbb{R})$ be two electric potentials. Let N be an open subset of S^{n-1} and consider F an open neighborhood of F_N, where F_N is defined as (1.1.2). If $\Lambda^\sharp_{A_1,q_1} = \Lambda^\sharp_{A_2,q_2}$ then $dA_1 = dA_2$ and $q_1 = q_2$ in Ω.*

Our next result concerns to prove a quantitative estimate of the identifiability result in Theorem 1.1.1. In this way and as is well known, in order to obtain stability results, one needs *a priori* bounds on the magnetic and electric potentials because one has to control oscillations. Hence, we introduce the class of admissible magnetic and electric potentials as follows.

Definition 1.1.2. Given $M > 0$ and $\gamma \in [0,1)$, we define the **class of admissible magnetic potentials** $\mathscr{A}(\Omega, M, \gamma)$ by

$$\mathscr{A}(\Omega, M, \gamma) = \left\{A \in C^{2,\gamma}(\overline{\Omega};\mathbb{R}^n) : \|A\|_{C^{2,\gamma}(\overline{\Omega})} \leqslant M\right\}.$$

Definition 1.1.3. Given $M > 0$ and $\sigma \in (0,1/2)$, we define the **class of admissible electric potentials** $\mathscr{Q}(\Omega, M, \sigma)$ by

$$\mathscr{Q}(\Omega, M, \sigma) = \left\{q \in L^\infty(\Omega;\mathbb{R}) : \|q\|_{L^\infty(\Omega)} + \|\chi_\Omega q\|_{H^\sigma(\mathbb{R}^n)} \leqslant M\right\}.$$

With these definitions at hand, we can now formulate our stability results.

Theorem 1.1.4. *Consider two positive constants M and $\gamma \in (0,1)$. Let N be an open subset of S^{n-1} and consider F an open neighborhood of F_N, where F_N is defined as (1.1.2). Then there exist $C > 0$ (depending on n, Ω, M, γ) and $\lambda \in (0, 1/2)$ (depending on n) such that the following estimate*

$$\|d(A_1 - A_2)\|_{L^2(\Omega)} \leqslant C \left|\log \left|\log \left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\|\right|\right|^{-\lambda/2},$$

holds true for all $A_1 \in \mathscr{A}(\Omega, M, \gamma)$ and $A_2 \in \mathscr{A}(\Omega, M, 0)$ satisfying $A_1 = A_2$ on $\partial\Omega$; and for all $q_1, q_2 \in L^\infty(\Omega)$.

Theorem 1.1.5. *Consider three positive constants M, $\sigma \in (0, 1/2)$ and $\gamma \in (0,1)$. Let N be an open subset of S^{n-1} and consider F an open neighborhood of F_N, where F_N is defined as (1.1.2). Then there exist $C > 0$ (depending on $n, \Omega, M, \sigma, \gamma$) and $\lambda \in (0, 1/2)$ (depending on n) such that the following estimate*

$$\|q_1 - q_2\|_{L^2(\Omega)} \leqslant C \left|\log \left|\log \left|\log \left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\|\right|\right|\right|^{-\lambda/2},$$

holds true for all $A_1 \in \mathscr{A}(\Omega, M, \gamma)$ and $A_2 \in \mathscr{A}(\Omega, M, 0)$ satisfying $A_1 = A_2$ on $\partial\Omega$; and all $q_1, q_2 \in \mathscr{Q}(\Omega, M, \sigma)$.

The proofs of Theorems 1.1.1, 1.1.4 and 1.1.5, will be carried out by proving an integral identity relating the partial boundary data, i.e. the partial DN maps, with the unknown magnetic and electric potentials in Ω, by means of solutions $u \in H^1(\Omega)$ of the magnetic Schrödinger equation $\mathcal{L}_{A,q} u = 0$ in Ω. In order to decode the information in the integral identity, we use special solutions, the so-called complex geometric optic solutions (CGO solutions). These solutions can be constructed by employing a suitable Carleman estimate with a linear weight. We use two kinds of CGO solutions. The first kind of CGO solutions will be obtained by using a suitable Carleman estimate with a linear weight and following the arguments used in [14] in order to have the required support constraint on the boundary, that is to construct solutions vanishing on $\partial\Omega \setminus B$. We emphasize that one of the main difficult parts to construct such required solutions is the derivation of a suitable Carleman estimate. In the literature, all previous results do not consider this support constraint on the boundary. For the second kind of solutions, we use the solutions constructed in [17], which do not need to have the support constraint. The combination of these solutions into the integral identity leads us to obtain Radon transforms, one for the difference of the magnetic fields $dA_1 - dA_2$, and another for the difference of the electric potentials $q_1 - q_2$. At this point, we use the injectivity on some suitable spaces of such Radon transforms to end up the proof of Theorem 1.1.1. We applied a quantitative estimate derived in [10] to the Radon transform of $dA_1 - dA_2$ to end up the proof of Theorem 1.1.4. The quantitative estimate involves a log of the difference of the DN maps. To prove Theorem 1.2.5, we applied the same quantitative estimate for the Radon transform now for $q_1 - q_2$, the Hodge decomposition derived by Tzou [50] (here we require the connectedness hypothesis) and the gauge invariance of the DN map in order to use the already established stability estimate for the magnetic fields. This step involves $\log\log$ of the difference of the partial DN maps, and by the quantitative estimate for the Radon transform, an extra logarithm has to be added.

1.2 Our second set of results. An IBVP with local data

In this section we state our second results. Now we consider an IBVP with local data associated to the magnetic Schrödinger equation (1.0.8). In contrast with the IBVP considered in the previous section and from the point of view of applications, to ask it seems to be more natural: is it still possible to recover information about the magnetic field and the electric potential from measurements on some subsets of the boundary for functions supported on the same subsets? We give a positive answer to this question under a geometric restriction over the domain Ω.

Suppose we divide the boundary $\partial\Omega$ of Ω into two subsets Γ_0 and $\Gamma := \partial\Omega \setminus \Gamma_0$. We shall call Γ_0 the inaccessible part of the boundary and Γ the accessible part . Before stating our second set of results, we first introduce the IBVP with local data under assumptions 1, 2 and 3 of the previous section.

Local data under smoothness

Under assumptions 1, 2 and 3 from the previous section, the global DN map (1.0.9) is well defined. Hence, we can define the local DNmap as

$$\begin{aligned} \Lambda^{\Gamma}_{A,q} : & H^{\frac{1}{2}}_{\Gamma}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega) \\ & f \to (\partial_\nu + iA\cdot\nu)u|_{\Gamma}, \end{aligned} \tag{1.2.1}$$

where $u \in H^1(\Omega)$ is the unique solution of (1.0.8) for a prescribed function $f \in H^{1/2}(\partial\Omega)$. Here, abusing the notation, $|_{\Gamma}$ denotes the local trace map of functions in $H^1(\Omega)$ onto the accessible part Γ of the boundary, that is, the restriction on Γ. The set $H^{\frac{1}{2}}_{\Gamma}(\partial\Omega)$ consists of all $f \in H^{\frac{1}{2}}(\partial\Omega)$ such that $\operatorname{supp} f \subset \overline{\Gamma}$. The latter condition will be described as "support constraint on Γ". The local boundary data $B^{\Gamma}_{A,q}$ is given by the graph of $\Lambda^{\Gamma}_{A,q}$:

$$B^{\Gamma}_{A,q} = \left\{ (u|_{\partial\Omega}, \Lambda^{\Gamma}_{A,q}(u|_{\partial\Omega})) \in H^{\frac{1}{2}}_{\Gamma}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) : u \in H^1(\Omega),\, \mathcal{L}_{A,q}u = 0 \right\}. \tag{1.2.2}$$

The proximity between two elements of the local Cauchy data sets $B^{\Gamma}_{A_1,q_1}$ and $B^{\Gamma}_{A_2,q_2}$, is naturally given by

$$\begin{aligned} & \left\| \left((u_1|_{\partial\Omega}, \Lambda^{\Gamma}_{A_1,q_1}(u_1|_{\partial\Omega})); (u_2|_{\partial\Omega}, \Lambda^{\Gamma}_{A_2,q_2}(u_2|_{\partial\Omega})) \right) \right\| \\ & := \|u_1 - u_2\|_{H^{\frac{1}{2}}_{\Gamma}(\partial\Omega)} + \left\| \Lambda^{\Gamma}_{A_1,q_1}(u_1|_{\partial\Omega}) - \Lambda^{\Gamma}_{A_2,q_2}(u_2|_{\partial\Omega}) \right\|_{H^{-\frac{1}{2}}(\partial\Omega)} \end{aligned} \tag{1.2.3}$$

This allows us to define a notion of distance between the local Cauchy data sets $B^{\Gamma}_{A_1,q_1}$ and $B^{\Gamma}_{A_2,q_2}$. In this regular framework, the IBVP with partial data consists of determining the magnetic potential A and the electric potential q from the knowledge of the local DN map $\Lambda^{\Gamma}_{A,q}$. In the absence of a magnetic potential, the identifiability result, (i.e. if $\Lambda^{\Gamma}_{0,q_1} = \Lambda^{\Gamma}_{0,q_2}$ then $q_1 = q_2$ in Ω) was obtained by Isakov [25] by assuming that the inaccessible part Γ_0 is either part of a hyperplane or part of a sphere. He proposed a reflection argument across the plane in order to construct special solutions vanishing on the inaccessible part of the boundary Γ. In the presence of a magnetic potential, the local DN map has the same gauge invariance like the global DN map, that is $\Lambda^{\Gamma}_{A,q} = \Lambda^{\Gamma}_{A+\nabla\varphi,q}$ for all $\varphi \in C^1(\overline{\Omega})$ with $\varphi|_{\partial\Omega} = 0$. Hence, for the identifiability problem, in this case we also expect to prove that $dA_1 = dA_2$ and $q_1 = q_2$ in Ω. Actually, it was proved by Krupchyk *et al* [27] assuming that Γ_0 is contained in a hyperplane, and that the magnetic potential A belongs to $W^{1,\infty}(\Omega)$ and the electric potential q belongs to $L^\infty(\Omega)$. In this context, our second results improve the aforementioned identifiability result and we also derive the corresponding stability estimates.

Our second results

Before stating our second results we introduce some assumptions, notations and definitions.

- ◇ **Assumption** 4. A priori, we do not assume any smoothness regularity over Ω and its boundary $\partial\Omega$, except that the inaccessible part of the boundary Γ_0 is contained in a hyperplane. More precisely:

$$\Omega \subset \{x \in \mathbb{R}^n \,:\, x_n > 0\} \quad \text{and} \quad \Gamma_0 = \partial\Omega \cap \{x \in \mathbb{R}^n \,:\, x_n = 0\} \neq \emptyset. \tag{1.2.4}$$

- ◇ **Assumption** 5. At the beginning, we assume that the magnetic potential A belongs to $L^\infty(\Omega;\mathbb{C}^n)$ and the electric potential q belongs to $L^\infty(\Omega;\mathbb{C})$.

We remark that assuming that zero is not a Dirichlet eigenvalue of $\mathcal{L}_{A,q}$ might be unnatural (see Assumption 3 in the previous section) because being an eigenvalue depends strongly on the coefficients of the operator $\mathcal{L}_{A,q}$, which at the same time depends on the magnetic potential A and the electric potential q, just the unknown parameters from which we want to obtain information. For this reason, we do not assume the nonzero Dirichlet eigenvalue for the magnetic Schrödinger operator $\mathcal{L}_{A,q}$.

In this lack of smoothness, we can not ensure, a priori, the existence and uniqueness of solutions for (C.1.40). As a consequence, the local DN map $\Lambda^{\Gamma}_{A,q}$ (see 1.2.1) is not well-defined and so we need to use instead the local linear map $N^{\Gamma}_{A,q}$. Also, since the magnetic potentials and Ω are not smooth, the local trace map $|_{\Gamma}$ of functions on the boundary has no sense as defined in (1.2.1) and so we extend its definition by the boundary local map T^{Γ}_r. Finally the boundary local data $B^{\Gamma}_{A,q}$ (see 1.2.2) will be replaced by the local Cauchy data set $C^{\Gamma}_{A,q}$, which is defined by

$$C^{\Gamma}_{A,q} = \left\{(T^{\Gamma}_r u, N^{\Gamma}_{A,q}(T^{\Gamma}_r u)) : u \in H^1(\Omega,\Gamma),\, \mathcal{L}_{A,q}u = 0 \text{ in } \Omega\right\}, \tag{1.2.5}$$

where, roughly speaking, the space $H^1(\Omega,\Gamma)$ denotes all the functions in $H^1(\Omega)$ vanishing on the inaccessible part of the boundary Γ_0. Since $N^{\Gamma}_{A,q}$ is not an operator anymore, the notion of distance between two elements of the local Cauchy data sets $C^{\Gamma}_{A_1,q_1}$ and $C^{\Gamma}_{A_2,q_2}$ as was defined in (1.2.3) has no sense. Instead of that, we introduce a notion of distance inspired by the Hausdorff distance and denoted by $dist(\,\cdot\,,\,\cdot\,)$. For expository convenience, the precise definition of the local boundary map T_r^{Γ}, the local linear map $N_{A,q}^{\Gamma}$, the space $H^1(\Omega,\Gamma)$ and $dist(\,\cdot,\,\cdot)$ will be given in Chapter 3. In this nonregular framework, we still have a gauge invariance for the local Cauchy data set $C_{A,q}^{\Gamma}$: if φ is a real-valued Lipschitz continuous function on $\overline{\Omega}$ with $\varphi|_{\partial\Omega}=0$, then $C_{A,q}^{\Gamma}=C_{A+\nabla\varphi,q}^{\Gamma}$, it can be deduced from Lemma 3.1 in [28]. Hence, for the identifiability problem we also expect to prove that $dA_1=dA_2$ and $q_1=q_2$ in Ω. More precisely, we give positive answers to the following questions:

- ⊛ **Identifiability**. Given two magnetic potentials A_1, A_2 and two electric potentials q_1, q_2 such that $C_{A_1,q_1}^{\Gamma}=C_{A_2,q_2}^{\Gamma}$, does it follow that $dA_1=dA_2$ and $q_1=q_2$ in Ω?
- ⊛ **Stability**. Does there exist a modulus of continuity m such that the following estimate:

$$\|dA_1-dA_2\|_{H^{-1}(\Omega)}+\|q_1-q_2\|_{H^{-1}(\Omega)}\leqslant m\left(\operatorname{dist}\left(C_{A_1,q_1}^{\Gamma},C_{A_2,q_2}^{\Gamma}\right)\right),$$

  holds true for all magnetic potentials A_1, A_2 and for all electric potentials q_1, q_2?

Our identifiability result is stated as follows.

Theorem 1.2.1. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set. Let $A_1, A_2\in L^{\infty}(\Omega;\mathbb{C}^n)$ be magnetic potentials and $q_1, q_2\in L^{\infty}(\Omega;\mathbb{C})$ be electric potentials. If $C_{A_1,q_1}^{\Gamma}=C_{A_2,q_2}^{\Gamma}$ then $dA_1=dA_2$ and $q_1=q_2$ in Ω.*

Our next result concerns to prove a quantitative estimate of the identifiability result in Theorem 1.2.1. In this way and as is well known, in order to obtain stability results, one needs *a priori* bounds on the magnetic and electric potentials because one has to control oscillations. At first, we introduce the so-called Besov spaces . Given $s>0$ we define the Besov space $B_s^{2,\infty}(\mathbb{R}^n;\mathbb{C})$ as the space consisting of all functions $f\in L^2(\mathbb{R}^n;\mathbb{C})$ for which the norm

$$\|f\|_{B_s^{2,\infty}(\mathbb{R}^n)}=\|f\|_{L^2(\mathbb{R}^n)}+\sup_{y\in\mathbb{R}^n\setminus\{0\}}\frac{\|f(\cdot+y)-f(\cdot)\|_{L^2(\mathbb{R}^n)}}{|y|^s} \tag{1.2.6}$$

is finite. Now we also introduce the class of admissible magnetic and electric potentials as follows.

Definition 1.2.2. Given $M>0$ and $s\in(0,1/2)$, we define the **class of admissible magnetic potentials** $\mathscr{A}(\Omega,M,s)$ as

$$\mathscr{A}(\Omega,M,s)=\left\{F\in L^{\infty}\cap B_s^{2,\infty}(\mathbb{R}^n;\mathbb{C}^n):\operatorname{supp}F\subset\overline{\Omega},\ \|F\|_{L^{\infty}\cap B_s^{2,\infty}}\leqslant M\right\},$$

Definition 1.2.3. Given $M>0$ and $s\in(0,1/2)$, we define the **class of admissible electric potentials** $\mathscr{Q}(\Omega,M,s)$ as

$$\mathscr{Q}(\Omega,M,s)=\left\{G\in L^{\infty}\cap B_s^{2,\infty}(\mathbb{R}^n;\mathbb{C}):\operatorname{supp}G\subset\overline{\Omega},\ \|G\|_{L^{\infty}\cap B_s^{2,\infty}}\leqslant M\right\}.$$

With these definitions at hand, we can now formulate our stability results.

Theorem 1.2.4 (Stability for the magnetic field). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Consider two constants $M > 0$ and $s \in (0, 1/2)$. Then there exist $C > 0$ (depending on $n, \Omega, M, s, \|q_1\|_{L^\infty}, \|q_2\|_{L^\infty}$) and an universal constant $\lambda \in (0,1)$ such that the following estimate*

$$\|d(A_1 - A_2)\|_{H^{-1}(\Omega)} \leqslant C \left|\log dist(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{\lambda}{n}s^2},$$

holds true for all $\chi_\Omega A_1, \chi_\Omega A_2 \in \mathscr{A}(\Omega, M, s)$ and for all $q_1, q_2 \in L^\infty(\Omega)$, whenever

$$dist\,(C_1^\Gamma, C_2^\Gamma) \leqslant e^{-C}.$$

Theorem 1.2.5 (Stability for the electric potential). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Consider two constants $M > 0$ and $s \in (0, 1/2)$. Then there exist $C > 0$ (depending on n, Ω, M, s) and a universal constant $\lambda \in (0, 1)$ such that the following estimate*

$$\|q_1 - q_2\|_{L^2(\Omega)} \leqslant C \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{\lambda}{n^2}s^3},$$

holds true for all $\chi_\Omega A_1$, $\chi_\Omega A_2 \in \mathscr{A}(\Omega, M, s)$ and for all $\chi_\Omega q_1$, $\chi_\Omega q_2 \in \mathscr{Q}(\Omega, M, s)$, whenever

$$dist\,(C_1^\Gamma, C_2^\Gamma) \leqslant e^{-C}.$$

Observe that in Theorems 1.2.4 and 1.2.5, we have imposed that the extensions by zero outside Ω of the magnetic and electric potentials belong to the Besov space $B_s^{2,\infty}$ with $s \in (0, 1/2)$. We mention that this is true for instance for Lipschitz domains, that is when $\partial\Omega$ is locally defined by the graph of a Lipschitz function, see [49]. In this case, it was proved by Faraco and Rogers that the characteristic function of Ω, χ_Ω, belongs to $H^{1/2-\varepsilon}(\mathbb{R}^n)$ for any $\varepsilon > 0$ small enough, see [19] for more details. This fact is our main motivation to consider in theorems 1.2.4 and 1.2.5, the Besov space's exponent s on $(0, 1/2)$.

The proofs of Theorems 1.2.1, 1.2.4 and 1.2.5, will be carried out by proving an integral identity relating the boundary data, i.e. the local Cauchy data sets, with the unknown magnetic and electric potentials in Ω, by means of solutions $u \in H^1(\Omega)$ for the magnetic Schrödinger equation $\mathcal{L}_{A,q}u = 0$. Such integral identity is an immediate consequence of the integral identity obtained in [8] in the context of full data case. In order to decode the information in the integral identity, we use special solutions having the vanishing condition on Γ_0. These solutions can be constructed by employing a Carleman estimate with a linear weight as in [28] and a reflection argument across Γ_0 as in [27]. The integral identity with these solutions leads us to obtain Fourier transforms, one for the difference between the magnetic fields $dA_1 - dA_2$ and another for the difference of the electric potentials $q_1 - q_2$, plus some error terms. At this point, we use the Riemann-Lebesgue lemma to control the error terms and the invertibility of the Fourier transforms in a suitable space to end up the proof of Theorem 1.2.1. We use a quantitative version of the Riemann-Lebesgue lemma derived in [24] and the Fourier transform of $dA_1 - dA_2$ to end up the proof of Theorem 1.2.4. To prove Theorem 1.2.5, we use the same quantitative version of the

Riemann-Lebesgue lemma, the Hodge decomposition derived in [8], the gauge invariance of the local Cauchy data sets in order to use the already established stability estimate for the magnetic fields and finally the Fourier transform for $q_1 - q_2$. Our stability results imply log-estimates for both magnetic and electric potentials. This is the best stability modulus that one can expect as was proved by Mandache [32] in the context of the DN map (1.0.9) and in the absence of a magnetic potential.

1.3 Bibliographical notes

In this section, we describe some of the earlier works related to the IBVPs treated in this dissertation.

1.3.1 Calderón problem

We start by describing the results concerning the Calderón problem. Here we assume that Ω is smooth enough. In 1980, Calderón proposed the study of the identifiability of the DN map Λ_σ defined by (1.0.2). He only obtained the identifiability for the linearized problem for constant conductivities [6] by using harmonic complex functions. As far as we known, the first result solving Calderón's problem was announced in 1985. It was obtained by Kohn and Vogelius [30] for piecewise real-analytic conductivities. A few years later, in 1987, Sylvester and Uhlmann [47] improved significantly the result for C^2-conductivities. Later on, several authors make progress in order to improve the C^2 assumption. In 1996, Brown [3] proved identifiability for $C^{\frac{3}{2}+\varepsilon}(\overline{\Omega})$-conductivities. Many years later, in 2012, Haberman and Tataru [22] improved the result for C^1-conductivities and for Lipschitz conductivities sufficiently close to the identity. Recently, the latter assumption was removed by Caro and Rogers [12] obtaining the result for arbitrary Lipschitz conductivities. Stability estimates with log-modulus of continuity were obtained by Alessandrini [1] in the context of [47]. Caro *et al* [9] extended the log-modulus of continuity for $C^{1,\varepsilon}$-conductivities. A reconstruction method was proposed by Nachman [34]. All previous identifiability results concerns the recovering of the conductivity in Ω. On the other hand, the identifiability on the boundary was obtained by Kohn and Vogelius [29] provided the conductivities are smooth near the boundary. This result was improved by Brown [4] for bounded conductivities. The ideas and techniques employed in the mentioned works were the basis of the development of anothers IBVPs. For instance, IBVPs associated to the elasticity equation [37]-[38], Dirac equation [43], Maxwell equation, among others.

1.3.2 IBVP associated to the magnetic Schrödinger operator $\mathcal{L}_{A,q}$

The main ideas of the foundational papers on Calderón's problem [6], [30] and [47] were to the study of IBVPs for the magnetic Schrödinger operator. Next, we outline the literature associated with these IBVPs.

IBVP associated to the magnetic Schrödinger operator $\mathcal{L}_{A,q}$ with full data

In the full data case and in the presence of a magnetic potential, special solutions adapted to the magnetic Schrödinger operator, were constructed by Sun [46] to prove identifiability for $\Lambda_{A,q}$, assuming smallness of the magnetic potential in a suitable space. In [36] the smallness was removed for C^2 and compactly supported magnetic potentials and L^∞ electrical potentials. Many efforts were put to improve the C^2 assumption on the magnetic potential. In this way, Tolmasky [48] obtained the result for $C^{\frac{2}{3}+\varepsilon}$ magnetic potentials. Salo [41] proved the result for a Dini continuous magnetic potential and also gave a proof for $C^{1+\varepsilon}$ with a reconstruction method [42]. Finally, the best identifiability result was obtained by Krupchyk and Uhlmann [28] for magnetic and electrical potentials in L^∞ and without any smoothness assumption over Ω. For this case, stability estimates were derived by Caro and Pohjola [8]. Almost all previous results were extended to the IBVP for $\mathcal{L}_{A,q}$ with partial and local boundary data. We will describe now these cases.

IBVP associated to the magnetic Schrödinger operator $\mathcal{L}_{A,q}$ with partial data

To describe the results for IBVPs with partial data, we divide the boundary $\partial\Omega$ in two open sets, F and B. Then, the partial DN map can by defined by

$$\begin{aligned} \Lambda_{A,q}^{B\to F} : & H_B^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega) \\ & f \to (\partial_\nu + iA\cdot\nu)u|_F, \end{aligned}$$

where ν is the exterior unit normal of $\partial\Omega$, the set $H_B^{\frac{1}{2}}(\partial\Omega)$ consists of all $f \in H^{\frac{1}{2}}(\partial\Omega)$ such that $\operatorname{supp} f \subset \overline{B}$, which we call support constraint on B; and $u \in H^1(\Omega)$ is the unique solution of the magnetic Schrödinger equation (1.0.8). Thus, according to the choice of the sets F and B, we can distinguish several types of partial data results.

In the absence of a magnetic potential ($A \equiv 0$), the pioneering work, which can be described as illuminating Ω from infinity, was obtained by Bukgheim and Uhlmann [5]. They considered a direction $\xi \in S^{n-1}$ and $F \subset \partial\Omega$ to be a neighborhood of the ξ-illuminated face or front region, defined as

$$\partial\Omega_{-,0}(\xi) = \{x \in \partial\Omega \;:\; \langle \xi, \nu(x)\rangle < 0\}. \tag{1.3.1}$$

For completeness we also define the ξ-shadowed face or back region as

$$\partial\Omega_{+,0}(\xi) = \{x \in \partial\Omega \;:\; \langle \xi, \nu(x)\rangle > 0\}. \tag{1.3.2}$$

In their work, they considered $B = \partial\Omega$. They obtained the identifiability result for $\Lambda_{0,q}^{\partial\Omega\to F}$. The corresponding stability estimates were derived by Heck and Wang [23]. Later, Kenig *et al* [26] obtained a similar identifiability result for $\Lambda_{0,q}^{B\to F}$ when F and B are neighborhoods of the respective illuminated and shadowed boundary regions of Ω from a point $x_0 \in \mathbb{R}^n$ (out of the convex hull of Ω), which were defined as

$$\partial\Omega_{-,0}(x_0) = \{x \in \partial\Omega \;:\; \langle x - x_0, \nu(x)\rangle < 0\} \tag{1.3.3}$$

and

$$\partial\Omega_{+,0}(x_0) = \{x \in \partial\Omega \,:\, \langle x - x_0, \nu(x)\rangle > 0\}\,,$$

respectively. In this case, if Ω is strictly convex then F could be arbitrarily small. To have an idea of the front and back sets in both of the previous cases, see Figure 1.1 below.

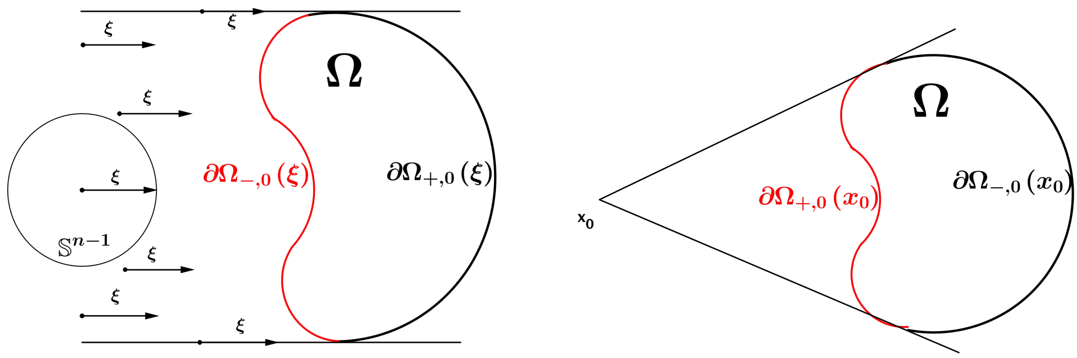

Figure 1.1: On the left, Ω is illuminated from a fixed direction $\xi \in S^{n-1}$. On the right, Ω is illuminated from a fixed point $x_0 \in \mathbb{R}^n$.

In the case of illumination from infinity, the supporting set B could also be restricted to a neighborhood of the shadowed region $\partial\Omega_{+,0}(\xi)$. In this case and in the absence of a magnetic potential $\Lambda_{0,q}^{B\to F}$, the corresponding stability estimates with the support constraint on B were derived by Caro, *et al* [10], by using a Radon transform. They also obtained stability estimates when illuminating Ω from a point but without the support constraint on B in [11], by using a geodesic ray transform on the sphere. In both cases, they obtained estimates with $\log\log$-modulus of continuity.

In the presence of a magnetic potential, the identifiability result for $\Lambda_{A,q}^{\partial\Omega\to F}$ (i.e., without the support constraint on B) in the case of illumination from a point $x_0 \in \mathbb{R}^n$ is due to Dos Santos Ferreira *et al* [17]. It was extended by Chung [14] to the case where the support constraint is on a neighborhood B of the shadowed boundary $\partial\Omega_{+,0}(x_0)$. Stability estimates for these results are still open and we are in working progress to obtain them.

To the best of our knowledge, the only stability estimates in the presence of a magnetic potential and for full data case was obtained by Tzou [50] for $\Lambda_{A,q}$. Moreover, he also obtained stability estimates for partial data from infinity without the support constraint on B, obtaining a $\log\log$-modulus of continuity.

In this context, one of our main contributions in this dissertation is the identifiability result, see Theorem 1.1.1; and its corresponding stability estimates, see Theorems 1.1.4 and 1.1.5 for the case of Bukhgeim and Uhlmann, that is illuminating Ω from the infinity, in the presence of a magnetic potential with the additional support constraint on B, a neighborhood of the shadowed boundary $\partial\Omega_{+,0}(\xi)$. As we have mentioned earlier, the main difficulty in proving these results was the construction of special solutions $u \in H^1(\Omega)$ for the magnetic Schrödinger equation $\mathcal{L}_{A,q}u = 0$ with the desired support constraint on B. The construction of these special solutions is given in Appendix C.

IBVP associated to the magnetic Schrödinger operator $\mathcal{L}_{A,q}$ with local data

The main key to face IBVPs with partial data is to derive suitable Carleman estimates. Most of them estimates has integral terms in Ω plus integral boundary errors coming from B and F. Roughly speaking, a Carleman estimate, see for instance (2.1.9), tell us that it is possible to bound the boundary error terms on $F \subset \partial\Omega$ by the boundedness of boundary error terms on B, and vice versa. Unfortunately these kinds of Carleman estimates can not be appplied for IBVPs with local data because we know the local DN map on the accessible part of the boundary Γ for functions supported on the same subset Γ. Hence the method of constructing special solutions by means of a suitable Carleman estimate with boundary terms does not work for IBVPs with local data. To overcome this obstruction, in the case of IVBP with local data and in the absence of a magnetic potential $\Lambda^{\Gamma}_{0,q}$, Isakov [25] obtained identifiability result by imposing that the inaccessible part of the boundary Γ_0 is either part of a hyperplane or part of a sphere in order to carry out a reflecting argument through the hyperplane. The sphere case can be reduced to the hyperplane case by means of a suitable transform. We explain briefly his ideas in the hyperplane case under Assumption 5, see (1.2.4). First he extended the electric potential q by zero outside Ω in $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$. We denote this extension by $\widetilde{q}$. Next, he made an even extension of $\widetilde{q}$ in $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n < 0\}$. Thus, at a first moment, he constructed special solutions $\widetilde{u} \in H^1(\mathbb{R}^n)$ of the equation $(-\Delta + \widetilde{q})\,\widetilde{u} = 0$ on the whole $\mathbb{R}^n$. By a straightforward computation, he proved that the function $u \in H^1(\mathbb{R}^n)$ defined by

$$u(x_1, \dots, x_n) = \widetilde{u}(x_1, \dots, x_n) - \widetilde{u}(x_1, \dots, -x_n)$$

satisfies $(-\Delta+\widetilde{q})u = 0$ in $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ and also satisfies $u|_{\{x_n=0\}} = 0$. Hence $u|_\Omega \in H^1(\Omega)$ is a solution of $(-\Delta + q)u = 0$ with $u|_{\Gamma_0} = 0$. In this case, stability estimates were derived by Heck and Wang [24]. Following Isakov's reflecting argument, Krupchyk *et al* [27] extended the identifiability result to the magnetic Schrödinger operator $\mathcal{L}_{A,q}$, assuming C^∞ boundary, magnetic potentials in $W^{1,\infty}$ and L^∞ electric potentials. Similar arguments were employed by Caro [7] to study an IBVP with local data for the Maxwell equation under the same flatness condition on Γ_0. Caro also obtained a *log*-stability estimate.

In this context, another contribution in this dissertation is to improve the latter identifiability result for magnetic and electric potentials both in L^∞. Furthermore, for the identifiability result, we are not assuming any smoothness condition for the boundary, except that the inaccessible part of the boundary also satisfies (1.2.4), see Theorem 1.2.1. We also obtain the corresponding stability estimates with log-modulus of continuity for the magnetic fields and the electric potentials, see Theorems 1.2.4 and 1.2.5. Our proofs follow arguments employed in [8] and [27].

Chapter 2

An IBVP for a magnetic Schrödinger operator with partial data

In this Chapter we prove the identifiability result stated in Theorem 1.1.1 and also the corresponding stability estimates for the magnetic fields and the electric potentials stated in the theorems 1.1.4 and 1.1.5. We start by proving an integral estimate relating the partial DN maps with the magnetic and electric potentials in Ω. Subsequently, in order to extract the information about $dA_1 - dA_2$ and $q_1 - q_2$ coded in the integral estimate we use two kinds of special solutions $u \in H^1(\Omega)$ for the magnetic Schrödinger equation $\mathcal{L}_{A,q}u = 0$. One with the desired vanishing condition on the compact subset of the boundary E and others constructed by Dos Santos Ferreira *et al.* which do not require the vanishing condition on E. This step leads us to obtain Radon transforms, one for the difference of the magnetic potentials $dA_1 - dA_2$ and other for the difference of the electric potentials $q_1 - q_2$. At this point, to end up the proof of Theorem 1.1.1 we use the invertibility of the Radon Transform. Finally, to end up the proof of theorems 1.1.4 and 1.1.5 we use a quantitative estimate for the Radon Transform derived in [10] plus a Hodge decomposition derived by Tzou [50] to the latter theorem.

Troughout this Chapter, unless otherwise indicated, we consider assumptions 1, 2 and 3 from Section 1.1 in Chapter 1. Also we shall denote by $\Lambda_i = \Lambda_{A_i,q_i}$ and $\Lambda_i^\sharp = \Lambda_{A_i,q_i}^\sharp$ to indicate the global and partial DN maps, respectively, with $i = 1, 2$.

2.1 Relating the DN maps with the magnetic and electric potentials in Ω

We state an integral estimate which involves a relation between the magnetic and electric potentials and the partial DN maps, see Proposition 2.1.6. We first recall the following lemma, proved in [17].

Lemma 2.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. If $A \in C^1(\overline{\Omega}; \mathbb{R}^n)$ and $q \in L^\infty(\Omega)$ then for all u, v in $L^2(\Omega)$ such that $\Delta u, \Delta v \in L^2(\Omega)$, we have the magnetic Green formula*

$$\begin{aligned}(2.1.1)\qquad &\langle \mathcal{L}_{A,q} u, v\rangle_{L^2(\Omega} - \langle u, \mathcal{L}_{A,\overline{q}} v\rangle_{L^2(\Omega)} \\ &\qquad = \langle u, (\partial_\nu + i\nu \cdot A)v\rangle_{L^2(\partial\Omega)} - \langle (\partial_\nu + i\nu\cdot A)u, v\rangle_{L^2(\partial\Omega)}.\end{aligned}$$

This lemma can be used to deduce an integral identity.

Lemma 2.1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. If $A_1, A_2 \in C^2(\overline{\Omega}; \mathbb{R}^n)$ and $q_1, q_2 \in L^\infty(\Omega)$ then*

$$\begin{aligned}(2.1.2)\qquad &\langle (\Lambda_1 - \Lambda_2)u_1, u_2\rangle_{L^2(\partial\Omega)} \\ &= \int_\Omega \left[(A_1 - A_2)\cdot(Du_1\overline{u}_2 + u_1\overline{Du}_2) + (A_1^2 - A_2^2 + q_1 - q_2)u_1\overline{u}_2\right],\end{aligned}$$

for all $u_1, u_2 \in H^1(\Omega)$ satisfying $\mathcal{L}_{A_1,q_1}u_1 = 0$ and $\mathcal{L}_{A_2,\overline{q}_2}u_2 = 0$ in Ω.

Proof. This lemma was implicitly established in section 4 of [17]. We shall repeat the proof because we will use some facts contained herein in the next sections. Let $u_1, u_2 \in H^1(\Omega)$ such that $\mathcal{L}_{A_1,q_1}u_1 = 0$ and $\mathcal{L}_{A_2,\overline{q}_2}u_2 = 0$ in Ω. We introduce an auxiliary function w satisfying

$$(2.1.3)\qquad \begin{cases} \mathcal{L}_{A_2,q_2} w = 0 & \text{in } \Omega, \\ w = u_1 & \text{on } \partial\Omega. \end{cases}$$

Thus, by the definition of the DN map, see (1.0.9), we get

$$\begin{aligned}(2.1.4)\qquad &\langle (\Lambda_1 - \Lambda_2)u_1, u_2\rangle_{L^2(\partial\Omega)} \\ &\quad = \langle (\partial\nu + i\nu\cdot A_1)u_1, u_2\rangle - \langle (\partial\nu + i\nu\cdot A_2)w, u_2\rangle_{L^2(\partial\Omega)} \\ &\quad = \langle \partial_\nu(u_1 - w) + i\nu\cdot(A_1 - A_2)u_1, u_2\rangle_{L^2(\partial\Omega)}.\end{aligned}$$

We now compute $\langle \mathcal{L}_{A_2,q_2}(w - u_1), u_2\rangle_{L^2(\Omega)}$ in two different ways. First, we use Lemma 2.1.1 and (2.1.3)-(2.1.4) to obtain

$$\begin{aligned}(2.1.5)\qquad &\langle \mathcal{L}_{A_2,q_2}(w - u_1), u_2\rangle_{L^2(\Omega)} \\ &= \left\langle w - u_1, \mathcal{L}_{A_2,\overline{q}_2}u_2\right\rangle_{L^2(\Omega)} + \langle w - u_1, (\partial_\nu + i\nu\cdot A_2)u_2\rangle_{L^2(\partial\Omega)} \\ &\qquad - \langle (\partial_\nu + i\nu\cdot A_2)(w - u_1), u_2\rangle_{L^2(\partial\Omega)} \\ &= \langle \partial_\nu(u_1 - w), u_2\rangle_{L^2(\partial\Omega)} \\ &= \langle (\Lambda_1 - \Lambda_2)u_1, u_2\rangle_{L^2(\partial\Omega)} - i\,\langle \nu\cdot(A_1 - A_2)u_1, u_2\rangle_{L^2(\partial\Omega)}.\end{aligned}$$

Again, from (2.1.3) and integration by parts we have

$$
\begin{aligned}
&\langle \mathcal{L}_{A_2,q_2}(w-u_1), u_2\rangle_{L^2(\Omega)} \\
&= \langle (\mathcal{L}_{A_1,q_1} - \mathcal{L}_{A_2,q_2})u_1, u_2\rangle_{L^2(\Omega)} \\
&= \left\langle (A_1 - A_2)\cdot Du_1 + D\cdot((A_1-A_2)u_1) + (A_1^2 - A_2^2 + q_1 - q_2)u_1, u_2\right\rangle_{L^2(\Omega)} \\
&= \int_\Omega \big((A_1 - A_2)\cdot(Du_1\overline{u}_2 + u_1\overline{Du}_2) + (A_1^2 - A_2^2 + q_1 - q_2)u_1\overline{u}_2\big)\, dx \\
&\qquad - i\,\langle \nu\cdot(A_1 - A_2)u_1, u_2\rangle_{L^2(\partial\Omega)}\,.
\end{aligned}
$$

We conclude the proof by combining this equality with (2.1.5). $\square$

Remark 2.1.3. *For technical reasons, we introduce some constants. Consider*

$$c = \sup_{x\in\Omega} |\xi\cdot x|\,, \quad \xi\in S^{n-1}, \tag{2.1.6}$$

(c is finite since Ω is bounded) and $\varepsilon > 0$ small enough such that

$$F_N \subset F_{N,\varepsilon} \subset\subset F, \tag{2.1.7}$$

where

$$F_{N,\varepsilon} = \bigcup_{\xi\in N} \Omega_{-,\varepsilon}(\xi) \tag{2.1.8}$$

where the set $\partial\Omega_{-,\varepsilon}(\xi)$ is defined in (1.1.1) and F_N in (1.1.2). In this setting, let $\chi \in C^\infty(\partial\Omega)$ be a cutoff function supported in F such that it equals to 1 on $F_{N,\varepsilon}$. Thus, from now on, we consider the partial DN map $\Lambda_i^\sharp = \chi\Lambda_i$ with $i = 1,2$. See (1.1.3).

Now the idea will be to use the identity (2.1.2) in order to obtain an integral inequality relating the partial DN maps with the unknown magnetic and electric potentials. As a first approach to do that, we obtain an integral estimate relating the difference of the global DN maps $\Lambda_1 - \Lambda_2$. The following Carleman estimate with boundary terms was derived by Dos Santos Ferreira *et al.*, see Proposition 2 in [17]. It will be useful to obtain bounds for $\Lambda_1^\sharp - \Lambda_2^\sharp$.

Proposition 2.1.4 (A Carleman estimate with boundary terms)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Let $\xi \in S^{n-1}$ and set $\varphi(x) = \xi\cdot x$. If $A \in C^1(\overline{\Omega};\mathbb{R}^n)$ and $q \in L^\infty(\Omega;\mathbb{R})$ then there exist two positive constants $\tau_0 > 0$ and $C > 0$ (both depending on $n, \Omega, \|A\|_{C^1}, \|q\|_{L^\infty}$) such that for all $u \in C^\infty(\overline{\Omega})\cap H_0^1(\Omega)$ the following estimate holds true for all $\tau \geqslant \tau_0$*

$$
\begin{aligned}
&\left\|\sqrt{\partial_\nu\varphi}\, e^{\tau\varphi}\partial_\nu u\right\|_{L^2(\Omega_{+,0}(\xi))} + \tau^{1/2}\left\|e^{\tau\varphi}u\right\|_{L^2(\Omega)} + \tau^{-1/2}\left\|e^{\tau\varphi}\nabla u\right\|_{L^2(\Omega)} \\
&\qquad \leqslant C\left(\tau^{-1/2}\left\|e^{\tau\varphi}\mathcal{L}_{A,q}u\right\|_{L^2(\Omega)} + \left\|\sqrt{-\partial_\nu\varphi}\, e^{\tau\varphi}\partial_\nu u\right\|_{L^2(\Omega_{-,0}(\xi))}\right),
\end{aligned} \tag{2.1.9}
$$

where ν denotes the exterior unit normal of $\partial\Omega$ and $\partial_\nu = \nu\cdot\nabla$. The sets $\partial\Omega_{\pm,0}(\xi)$ are defined by (1.3.1)-(1.3.2).

Remark 2.1.5. *The Carleman estimate (2.1.9) is still true for all u in $H_0^1(\Omega)$ such that $\mathcal{L}_{A,q}u \in L^2(\Omega)$. This could be seen by a standard regularization method. The vanishing of the trace of the function u is essential for this estimate. Notice that in the above inequality we bound the $L^2(\Omega_{+,0}(\xi))$-norm by the $L^2(\Omega_{-,0}(\xi))$-norm plus remainder terms in $L^2(\Omega)$-norm. In other words, we bound the unknown measurements of the shadow face of $\partial\Omega$ by know measurements of the illuminated face but we have to pay with remainder terms in $L^2(\Omega)$-norm. This estimate will be useful in our approach.*

Proposition 2.1.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$. Consider the two positive constants, c given by (2.1.6) and ε satisfy (2.1.7). Let $M > 0$ and $\gamma \in (0,1)$. Consider $A_1 \in \mathscr{A}(\Omega, M, \gamma)$, $A_2 \in \mathscr{A}(\Omega, M, 0)$ and $q_1, q_2 \in L^\infty(\Omega)$. If $u_1, u_2 \in H^1(\Omega)$ such that $\mathcal{L}_{A_1,q_1}u_1 = 0$ and $\mathcal{L}_{A_2,\overline{q}_2}u_2 = 0$ both in Ω then there exist two positive constants τ_0 and C (both depending on $n, \Omega, M, \gamma, \varepsilon$) such that the estimate*

$$
\begin{aligned}
&\left|\langle(\Lambda_1 - \Lambda_2)u_1, u_2\rangle_{L^2(\partial\Omega)}\right| \\
&\leqslant C\,\|\chi(\Lambda_1 - \Lambda_2)\|\Big(\|u_1\|_{H^1(\Omega)}\|u_2\|_{H^1(\Omega)} \\
&\qquad\qquad + e^{\tau c}\|u_1\|_{H^1(\Omega)}\left\|e^{\tau\xi\cdot x}u_2\right\|_{L^2(\partial\Omega)}\Big) \\
&\quad + C\tau^{-\frac{1}{2}}\left\|e^{-\tau\xi\cdot x}(\mathcal{L}_{A_1,q_1} - \mathcal{L}_{A_2,q_2})u_1\right\|_{L^2(\Omega)}\left\|e^{\tau\xi\cdot x}u_2\right\|_{L^2(\partial\Omega)} \\
&\quad + C\left\|e^{-\tau\xi\cdot x}u_1\right\|_{L^2(\partial\Omega)}\left\|e^{\tau\xi\cdot x}u_2\right\|_{L^2(\partial\Omega)}
\end{aligned} \tag{2.1.10}
$$

holds true for all $\tau \geqslant \tau_0$ and for all $\xi \in N$.

Remark 2.1.7. *The set $\mathcal{A}(\Omega, M, \gamma)$ with $0 \leqslant \gamma < 1$, represents the class of admissible magnetic potential, see Definition 1.1.2.*

Proof. We begin by denoting $\Lambda_{A_i,q_i} = \Lambda_i$ for $i = 1, 2$. Let us decompose the difference between the DN maps in the following way

$$\Lambda_1 - \Lambda_2 = \chi(\Lambda_1 - \Lambda_2) + (1 - \chi)(\Lambda_1 - \Lambda_2).$$

Thus, we have

$$
\begin{aligned}
\langle(\Lambda_1 - \Lambda_2)u_1, u_2\rangle_{L^2(\partial\Omega)} &= \langle\chi(\Lambda_1 - \Lambda_2)u_1, u_2\rangle_{L^2(\partial\Omega)} \\
&\quad + \langle(1 - \chi)(\Lambda_1 - \Lambda_2)u_1, u_2\rangle_{L^2(\partial\Omega)}.
\end{aligned} \tag{2.1.11}
$$

We now estimate each term of the right-hand side of the previous identity. By using Cauchy-Schwarz inequality, the first term can be estimated as follows

$$
\begin{aligned}
\left|\int_{\partial\Omega}\chi(\Lambda_1 - \Lambda_2)u_1\overline{u}_2 dS\right| &\leqslant \|\chi(\Lambda_1 - \Lambda_2)\|\,\|u_1\|_{H^{\frac{1}{2}}(\partial\Omega)}\|u_2\|_{H^{\frac{1}{2}}(\partial\Omega)} \\
&\leqslant \|\chi(\Lambda_1 - \Lambda_2)\|\,\|u_1\|_{H^1(\Omega)}\|u_2\|_{H^1(\Omega)}.
\end{aligned} \tag{2.1.12}
$$

To estimate the second term, we shall require a more refined analysis. Let w be a function such that it satisfies (2.1.3). Then for every $\xi \in N$ we get

$$
\begin{aligned}
&\left|\int_{\partial\Omega}(1-\chi)(\Lambda_1-\Lambda_2)u_1\overline{u}_2 dS\right| \\
&=\left|\int_{\Omega_{-,\varepsilon}(\xi)\cup(\partial\Omega\setminus\Omega_{-,\varepsilon}(\xi))}(1-\chi)(\Lambda_1-\Lambda_2)u_1\overline{u}_2 dS\right| \\
&=\left|\int_{\partial\Omega\setminus\Omega_{-,\varepsilon}(\xi)}(1-\chi)(\Lambda_1-\Lambda_2)u_1\overline{u}_2 dS\right| \\
&\leqslant C_1\left\|e^{-\tau\xi\cdot x}(\Lambda_1-\Lambda_2)u_1\right\|_{L^2(\partial\Omega\setminus\Omega_{-,\varepsilon}(\xi))}\left\|e^{\tau\xi\cdot x}u_2\right\|_{L^2(\partial\Omega\setminus\Omega_{-,\varepsilon}(\xi))}.
\end{aligned}
\tag{2.1.13}
$$

We next turn to the $L^2(\partial\Omega\setminus\Omega_{-,\varepsilon}(\xi))$-norms in the previous inequality. Since $u_1\in H^1(\Omega)$ and $\mathcal{L}_{A_2,q_2}(w-u_1)=(\mathcal{L}_{A_1,q_1}-\mathcal{L}_{A_2,q_2})u_1$, where w is the auxiliary function in (2.1.3), it follows that $\mathcal{L}_{A_2,q_2}(w-u_1)\in L^2(\Omega)$. Moreover, we have that $w-u_1\in H_0^1(\Omega)$. Hence, the Carleman estimate (2.1.9) from Proposition 2.1.4 and Remark 2.1.5 imply that

$$
\begin{aligned}
&\left\|e^{-\tau\xi\cdot x}(\Lambda_1-\Lambda_2)u_1\right\|_{L^2(\partial\Omega\setminus\Omega_{-,\varepsilon}(\xi))} \\
&=\left\|e^{-\tau\xi\cdot x}\left(\partial_\nu(u_1-w)+i\nu\cdot(A_1-A_2)u_1\right)\right\|_{L^2(\partial\Omega\setminus\Omega_{-,\varepsilon}(\xi))} \\
&\leqslant\left\|e^{-\tau\xi\cdot x}\partial_\nu(u_1-w)\right\|_{L^2(\partial\Omega\setminus\Omega_{-,\varepsilon}(\xi))}+C_1\left\|e^{-\tau\xi\cdot x}u_1\right\|_{L^2(\partial\Omega\setminus\Omega_{-,\varepsilon}(\xi))} \\
&\leqslant\frac{1}{\sqrt{\varepsilon}}\left\|\sqrt{\langle\xi\cdot\nu(\cdot)\rangle}e^{-\tau\xi\cdot x}\partial_\nu(u_1-w)\right\|_{L^2(\partial\Omega\setminus\Omega_{-,\varepsilon}(\xi))} \\
&\qquad+C_1\left\|e^{-\tau\xi\cdot x}u_1\right\|_{L^2(\partial\Omega\setminus\Omega_{-,\varepsilon}(\xi))} \\
&\leqslant\frac{1}{\sqrt{\varepsilon}}\left\|\sqrt{\langle\xi\cdot\nu(\cdot)\rangle}e^{-\tau\xi\cdot x}\partial_\nu(u_1-w)\right\|_{L^2(\Omega_{+,0}(\xi))} \\
&\qquad+C_1\left\|e^{-\tau\xi\cdot x}u_1\right\|_{L^2(\partial\Omega)} \\
&\leqslant\frac{C_2}{\sqrt{\varepsilon}}\Bigg(\left\|\sqrt{-\langle\xi\cdot\nu(\cdot)\rangle}e^{-\tau\xi\cdot x}\partial_\nu(u_1-w)\right\|_{L^2(\partial\Omega_{-,0}(\xi))} \\
&\quad+\tau^{-\frac{1}{2}}\left\|e^{-\tau\xi\cdot x}\mathcal{L}_{A_2,q_2}(w-u_1)\right\|_{L^2(\Omega)}\Bigg)+C_1\left\|e^{-\tau\xi\cdot x}u_1\right\|_{L^2(\partial\Omega)} \\
&\leqslant\frac{C_2}{\sqrt{\varepsilon}}\Bigg(\left\|e^{-\tau\xi\cdot x}\partial_\nu(u_1-w)\right\|_{L^2(\partial\Omega_{-,0}(\xi))} \\
&\quad+\tau^{-\frac{1}{2}}\left\|e^{-\tau\xi\cdot x}(\mathcal{L}_{A_1,q_1}-\mathcal{L}_{A_2,q_2})u_1\right\|_{L^2(\Omega)}\Bigg)+C_1\left\|e^{-\tau\xi\cdot x}u_1\right\|_{L^2(\partial\Omega)}.
\end{aligned}
\tag{2.1.14}
$$

By using (2.1.4), we estimate the $L^2(\partial\Omega_{-,0}(\xi))$-norm in the last inequality as follows:

$$
\begin{aligned}
&\left\| e^{-\tau\xi\cdot x}\partial_\nu(u_1 - w)\right\|_{L^2(\partial\Omega_{-,0}(\xi))} \\
&= \left\| e^{-\tau\xi\cdot x}\left[(\Lambda_1 - \Lambda_2)u_1 - i\nu\cdot(A_1 - A_2)u_1\right]\right\|_{L^2(\partial\Omega_{-,0}(\xi))} \\
&\leqslant \left\| e^{-\tau\xi\cdot x}\chi(\Lambda_1 - \Lambda_2)u_1\right\|_{L^2(\partial\Omega)} \\
&\qquad + \left\| e^{-\tau\xi\cdot x} i\nu\cdot(A_1 - A_2)u_1\right\|_{L^2(\partial\Omega_{-,0}(\xi))} \\
&\leqslant e^{\tau k}\left\|\chi(\Lambda_1 - \Lambda_2)\right\|\left\|u_1\right\|_{H^{\frac{1}{2}}(\partial\Omega)} + C_3\left\| e^{-\tau\xi\cdot x}u_1\right\|_{L^2(\partial\Omega)}.
\end{aligned}
\tag{2.1.15}
$$

Thus, replacing (2.1.14) and (2.1.15) into (2.1.13) gives us

$$
\begin{aligned}
&\left|\int_{\partial\Omega}(1-\chi)(\Lambda_1 - \Lambda_2)u_1\overline{u}_2 dS\right| \\
&\leqslant C_4\Big(\varepsilon^{-1/2}e^{\tau k}\left\|\chi(\Lambda_1 - \Lambda_2)\right\|\left\|u_1\right\|_{H^1(\Omega)} \\
&\qquad\qquad + \varepsilon^{-1/2}\tau^{-\frac{1}{2}}\left\| e^{-\tau\xi\cdot x}(\mathcal{L}_{A_1,q_1} - \mathcal{L}_{A_2,q_2})u_1\right\|_{L^2(\Omega)} \\
&\qquad\qquad\qquad + \left\| e^{-\tau\xi\cdot x}u_1\right\|_{H^1(\Omega)}\Big)\left\| e^{\tau\xi\cdot x}u_2\right\|_{L^2(\partial\Omega)}.
\end{aligned}
\tag{2.1.16}
$$

Finally, we conclude the proof by replacing (2.1.12) and (2.1.16) into (2.1.11). □

2.2 Construction of special solutions - CGO solutions

We would like to replace $\Lambda_1^\sharp - \Lambda_2^\sharp$ instead of $\chi(\Lambda_1 - \Lambda_2)$ in the right-hand side of (2.1.10). By definition of $\Lambda_1^\sharp - \Lambda_2^\sharp$, see (1.1.3), this can be done by providing that the solution $u_1 \in H^1(\Omega)$ of $\mathcal{L}_{A_1,q_1}u_1 = 0$ satisfies the additional support constraint $\operatorname{supp} u_1 \subset \overline{B}$. Recall the B is a open neighborhood of the set B_N, which in turn is an open neighborhood of the shadowed face of $\partial\Omega$, see (1.1.2). On the other hand, notice that u_2 do not require this supporting condition on B. To achieve the supporting condition for u_1 we consider the set Z_N defined by

$$Z_N = \bigcup_{\xi\in N}\{x \in \partial\Omega \ : \ \langle\xi, \nu(x)\rangle = 0\}$$

and let E be a compact subset of $\partial\Omega$ such that

$$\partial\Omega \setminus B \subset E \subset F_N \setminus Z_N, \tag{2.2.1}$$

where the set F_N is defined in (1.1.2). Observe that if $u_1|_E = 0$ then $\operatorname{supp} u_1 \subset \overline{B}$. As far as we know, for the case when Ω is illuminated from infinity and in the presence of a magnetic potential, has still not been constructed solutions u_1 with the desired support constraint on B. The only result is for the case when Ω is illuminated from a fixed point. This case was proved by Chung [14]. Following Chung's ideas, we obtain the following theorem whose proof is given on the Appendix C for expository convenience.

Theorem 2.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$. Let $\xi, \zeta \in S^{n-1}$ be a pair of orthonormal vectors and let E satisfies (2.2.1). Consider $\gamma \in (0,1)$. If $A_1 \in C^{2,\gamma}(\overline{\Omega}; \mathbb{R}^n)$ and $q_1 \in L^\infty(\Omega)$, then there exist three positive constants: τ_0, C (both depending on $n, \Omega, \|A_1\|_{C^{2,\gamma}}, \|q_1\|_{L^\infty}$) and $\underline{\gamma}$ (depending on n) with $0 < \underline{\gamma} < \gamma$ such that the equation*

$$\begin{cases} \mathcal{L}_{A_1,q_1} u = 0 & \text{in } \Omega \\ u|_E = 0 \end{cases}$$

has a solution $u_1 \in H^1(\Omega)$ of the form

$$u_1 = e^{\tau(\xi\cdot x + i\zeta\cdot x)}\left(e^{\Phi_1} + r_1\right) - e^{\tau l} b,$$

with the following properties:

(i) The function $\Phi_1 \in C^{3,\underline{\gamma}}(\Omega)$ satisfies in Ω

$$(\xi + i\zeta)\cdot\nabla\Phi_1 + i(\xi + i\zeta)\cdot A_1 = 0, \tag{2.2.2}$$

$$\|\Phi_1\|_{W^{\alpha,\infty}} \leqslant C\,\|A_1\|_{C^\alpha(\Omega)}\ ,\ |\alpha| \leqslant 2. \tag{2.2.3}$$

and

$$\|\Phi_1\|_{C^{3,\underline{\gamma}}(\Omega)} \leqslant C\,\|A_1\|_{C^{2,\gamma}(\Omega)}\,. \tag{2.2.4}$$

(ii) The function l depends on the a priori bounds of A_1 and q_1, and satisfies

$$\Re l(x) = \xi\cdot x - k(x),$$

where $k(x) \simeq dist(x, E)$ in G, a neighborhood of E on $\mathbb{R}^n$.

(iii) The function b belongs to $C^{1,\underline{\gamma}}(\Omega)$ with $\operatorname{supp} b \subset G$; and it depends on the a priori bounds of A_1 and q_1.

(iv) Finally, $r_1 \in H^1(\Omega)$ satisfies $r|_E = 0$ and for all $\tau \geqslant \tau_0$ the following estimates hold true

$$\|\partial^\alpha r_1\|_{L^2(\Omega)} \leqslant C\tau^{|\alpha|-1}, |\alpha| \leqslant 1,$$
$$\|r\|_{L^2(\partial\Omega)} \leqslant C\tau^{-1/2}.$$

Moreover, we have

$$\|l\|_{H^1(\Omega)} \leqslant C, \quad \|b\|_{H^1(\Omega)} \leqslant C \tag{2.2.5}$$

and

$$\left\|e^{-\tau k}\right\|_{L^2(\Omega)} \leqslant C\tau^{-1/2}, \quad \left\|e^{-\tau k}\right\|_{L^\infty(\Omega)} \leqslant C. \tag{2.2.6}$$

On the other hand, for solutions $u_2 \in H^1(\Omega)$ of the equation $\mathcal{L}_{A_2,\overline{q_2}} u_2 = 0$, we will use the special solutions already constructed by Dos Santos Ferreira *et al.*, see Lemma 3.4 in [17]. These solutions do not require the support constraint on E.

Theorem 2.2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Let $\xi, \zeta \in S^{n-1}$ be a pair of orthonormal vectors. If $A_2 \in C^2(\overline{\Omega}; \mathbb{R}^n)$ and $q_2 \in L^\infty(\Omega)$ then there exist two positive constants τ_0 and C (both depending on $n, \Omega, \|A_2\|_{C^2}, \|q_2\|_{L^\infty}$) such that the equation $\mathcal{L}_{A_2,\overline{q}_2} u = 0$ has a solution $u_2 \in H^1(\Omega)$ of the form*

$$u_2 = e^{-\tau(\xi\cdot x - i\zeta\cdot x)}\left(e^{\Phi_2} g + r_2\right),$$

with the following properties:

(i) The function Φ_2 satisfies in Ω

$$(\xi + i\zeta)\cdot\nabla\overline{\Phi}_2 - i(\xi + i\zeta)\cdot A_2 = 0. \tag{2.2.7}$$

and

$$\|\Phi_2\|_{W^{\alpha,\infty}} \leqslant C\,\|A_2\|_{C^\alpha(\Omega)}\ ,\ |\alpha| \leqslant 2. \tag{2.2.8}$$

(ii) The function g is smooth and satisfies in Ω

$$(\xi + i\zeta)\cdot\nabla g = 0. \tag{2.2.9}$$

(iii) The function r_2 belongs to $H^1(\Omega)$ and satisfies the following estimate

$$\|\partial^\alpha r_2\|_{L^2(\Omega)} \leqslant C\tau^{|\alpha|-1}\,\|g\|_{H^2(\Omega)}\,, \quad |\alpha| \leqslant 1,$$

for all $\tau \geqslant \tau_0$.

Remark 2.2.3. *We mention that Theorem 2.2.2 was stated for a general limiting Carleman weight (LCW) φ instead of the linear $\xi\cdot x$. Moreover, if φ is a LCW then $-\varphi$ is also a LCW. As a consequence, the theorems 2.2.1 and 2.2.2 remain true replacing $\xi\cdot x$ by $-\xi\cdot x$; and in this case we have analogous estimates for the respective solutions u_1 and u_2. For precise definition of LCW, see Appendix A.*

Corollary 2.2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$. Let $M > 0$ and $\gamma \in (0,1)$. Consider $A_1 \in \mathscr{A}(\Omega, M, \gamma)$, $A_2 \in \mathscr{A}(\Omega, M, 0)$ and $q_1, q_2 \in L^\infty(\Omega)$. Let $u_1 \in H^1(\Omega)$ be a solution of $\mathcal{L}_{A_1,q_1} u = 0$ constructed in Theorem C.1.1 and let $u_2 \in H^1(\Omega)$ be a solution of $\mathcal{L}_{A_2,\overline{q_2}} u = 0$ constructed in Theorem 2.2.2. Then there exist $\tau_0 > 0$ and $C > 0$ (both depending on n, Ω, M, γ) such that the estimate*

$$\begin{aligned} &\tau^{-1}\left|\langle(\Lambda_1 - \Lambda_2)u_1, u_2\rangle_{L^2(\partial\Omega)}\right| \\ &\qquad \leqslant C\left(e^{4\tau c}\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\| + \tau^{-\frac{1}{2}}\right)\|\overline{g}\|_{H^2(\Omega)}\,, \end{aligned} \tag{2.2.10}$$

holds true for all $\tau \geqslant \tau_0$.

Proof. We start by computing the norms corresponding to u_1 in the right hand side of (2.1.10). The estimates for u_2 are similar. By Theorem C.1.1, the function u_1 has the form

$$u_1 = e^{\tau(\xi\cdot x+\zeta\cdot x)}\left(e^{\Phi_1}+r_1\right) - e^{\tau l}b$$

and there exist two positive constants C_1 and τ_1 such that the following estimate

$$\|\partial^\alpha r_1\|_{L^2(\Omega)} \leqslant C_1\tau^{|\alpha|-1}, \quad |\alpha|\leqslant 1, \tag{2.2.11}$$

holds true for all $\tau \geqslant \tau_1$. Also, we have the estimate

$$\left\|e^{\tau l}\right\|_{L^\infty(\Omega)} = \left\|e^{\tau(\xi\cdot x-k(x))}\right\|_{L^\infty(\Omega)} \leqslant \left\|e^{\tau\xi\cdot x}\right\|_{L^\infty(\Omega)} \leqslant e^{\tau c}.$$

For convenience, we denote

$$a_1 = e^{\Phi_1}, \varphi(x) = \xi\cdot x, \psi(x) = \zeta\cdot x. \tag{2.2.12}$$

Since $\operatorname{Re} l(x) = \xi\cdot x - k(x)$, the above estimates and (C.1.6) imply that

$$\begin{aligned}
&\|u_1\|_{H^1(\Omega)} = \|u_1\|_{L^2(\Omega)} + \|\nabla u_1\|_{L^2(\Omega)}\\
&= \left\|e^{\tau(\varphi+i\psi)}(a_1+r_1) - e^{\tau l}b\right\|_{L^2(\Omega)}\\
&\quad + \left\|\tau\nabla(\varphi+i\psi)e^{\tau(\varphi+i\psi)}(a_1+r_1)\right\|_{L^2(\Omega)}\\
&\quad + \left\|e^{\tau(\varphi+i\psi)}(\nabla a_1+\nabla r_1) - \nabla(e^{\tau l}b)\right\|_{L^2(\Omega)}\\
&\leqslant \left\|e^{\tau(\varphi+i\psi)}(a_1+r_1)\right\|_{L^2(\Omega)} + \left\|e^{\tau l}b\right\|_{L^2(\Omega)}\\
&\quad + \left\|\tau b e^{\tau l}\nabla l + e^{\tau l}\nabla b\right\|_{L^2(\Omega)}\\
&\quad + \left\|\tau\nabla(\varphi+i\psi)e^{\tau(\varphi+i\psi)}(a_1+r_1) + e^{\tau(\varphi+i\psi)}(\nabla a_1+\nabla r_1)\right\|_{L^2(\Omega)}\\
&\leqslant C_1\left\|e^{\tau\varphi}\right\|_{L^\infty(\Omega)}\|a_1+r_1\|_{L^2(\Omega)} + \left\|e^{\tau l}\right\|_{L^\infty(\Omega)}\|b\|_{L^2(\Omega)}\\
&\quad + \tau\left\|e^{\tau l}\right\|_{L^\infty(\Omega)}\|b\|_{H^1(\Omega)} + C_1\tau\left\|e^{\tau\varphi}\right\|_{L^\infty(\Omega)}\|a_1+r_1\|_{L^2(\Omega)}\\
&\quad + C_1\left\|e^{\tau\varphi}\right\|_{L^\infty(\Omega)}\|\nabla(a_1+r_1)\|_{L^2(\Omega)}\\
&\leqslant C_2\tau e^{\tau c}\|a_1+r_1\|_{H^1(\Omega)} + C_2\tau e^{\tau c}\|b\|_{H^1(\Omega)} \leqslant C_3\tau e^{\tau c}.
\end{aligned} \tag{2.2.13}$$

We continue in this fashion to compute

$$\begin{aligned}
\left\|e^{-\tau\varphi}u_1\right\|_{L^2(\partial\Omega)} &= \left\|e^{-i\tau\psi}(a_1+r_1) + e^{-\tau\varphi}e^{\tau l}b\right\|_{L^2(\partial\Omega)}\\
&\leqslant \|a_1+r_1\|_{L^2(\partial\Omega)} + \left\|e^{-\tau k(x)}b\right\|_{L^2(\partial\Omega)}\\
&\leqslant \|a_1+r_1\|_{H^1(\Omega)} + \|b\|_{H^1(\Omega)} \leqslant C_4.
\end{aligned} \tag{2.2.14}$$

Finally by denoting $V = a_1 + r_1 + e^{-\tau(\varphi+i\psi)}e^{\tau l}b$, we get

$$\begin{aligned}
&\left\|e^{-\tau(\varphi+i\psi)}(\mathcal{L}_{A_1,q_1} - \mathcal{L}_{A_2,q_2})u_1\right\|_{L^2(\Omega)} \\
&= \left\|e^{-\tau(\varphi+i\psi)}(\mathcal{L}_{A_1,q_1} - \mathcal{L}_{A_2,q_2})\left[e^{\tau(\varphi+i\psi)}V\right]\right\|_{L^2(\Omega)} \\
&\leqslant \|(A_1 - A_2)\cdot[\tau(\nabla\varphi + i\nabla\psi)V]\|_{L^2(\Omega)} \\
&\qquad + \|(A_1 - A_2)\cdot\nabla V\|_{L^2(\Omega)} \\
&\qquad\quad + \|\nabla\cdot(A_1 - A_2)V\|_{L^2(\Omega)} \\
&\qquad\qquad + \left\|(A_1^2 - A_2^2 + q_1 - q_2)V\right\|_{L^2(\Omega)} \\
&\leqslant C_6\left(\tau\|a_1 + r_1 + b\|_{L^2(\Omega)} + \|\nabla(a_1 + r_1 + b)\|_{L^2(\Omega)}\right) \\
&\leqslant C_6\tau\|a_1 + r_1 + b\|_{H^1(\Omega)} \leqslant C_7\tau.
\end{aligned} \tag{2.2.15}$$

On the other hand, by Theorem 2.2.2 the function u_2 has the form

$$u_2 = e^{-\tau(\xi\cdot x - i\zeta\cdot x)}\left(e^{\Phi_2}g + r_2\right)$$

and there exist two positive constants C_2 and τ_2 such that the following estimate

$$\|\partial^\alpha r_2\|_{L^2(\Omega)} \leqslant C_2\tau^{|\alpha|-1}\|\overline{g}\|_{H^2(\Omega)}, \quad |\alpha| \leqslant 1, \tag{2.2.16}$$

holds true for all $\tau \geqslant \tau_2$. The above inequality and analogous arguments as it was employed for the boundedness of u_1, gives us the following estimates for u_2

$$\|u_2\|_{H^1(\Omega)} \leqslant C_4\tau e^{\tau c}\|\overline{g}\|_{H^2(\Omega)} \tag{2.2.17}$$

and

$$\left\|e^{\tau\xi\cdot x}u_2\right\|_{L^2(\partial\Omega)} \leqslant C_5\|\overline{g}\|_{H^2(\Omega)}. \tag{2.2.18}$$

Thus, by combining the estimates (2.2.13)-(2.2.15) into (2.1.10), and taking into account that there exists $\tau_3 > 0$ such that $\tau \leqslant e^{2\tau c}$ for all $\tau \geqslant \tau_3$, we get

$$\begin{aligned}
&\left|\langle(\Lambda_1 - \Lambda_2)u_1, u_2\rangle_{L^2(\partial\Omega)}\right| \\
&\leqslant C\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\|\left(\tau^2 e^{2\tau c} + \tau e^{2\tau c}\right)\|\overline{g}\|_{H^2(\Omega)} \\
&\qquad + C\left(\tau^{1/2} + 1\right)\|\overline{g}\|_{H^2(\Omega)} \\
&\leqslant C\left(\tau e^{4\tau c}\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\| + \tau^{1/2}\right)\|\overline{g}\|_{H^2(\Omega)}.
\end{aligned}$$

Finally, by multiplying by τ^{-1} both sides of the previous inequality and taking $\tau_0 = \max(\tau_1, \tau_2, \tau_3)$, we conclude the proof. □

2.3 Proof of log-type estimate for the magnetic fields

Corollary 2.2.4 gives us an estimate for the left-hand side of the identity (2.1.2). The task now is to estimate the right-hand side multiplied by τ^{-1}, that is to estimate the expression

$$\int_{\Omega} \left[(A_1 - A_2)\cdot(\tau^{-1}Du_1\overline{u}_2 + \tau^{-1}u_1\overline{Du_2}) + \tau^{-1}(A_1^2 - A_2^2 + q_1 - q_2)u_1\overline{u}_2\right] dx.$$

For convenience, we denote $\rho(x) = (\xi + i\zeta)\cdot x$, $a_1 = e^{\Phi_1}$, $u_r = e^{\tau(-\rho+l)}b$ and $a_2 = e^{\Phi_2}g$. Hence, the solutions u_1 and u_2 constructed in theorems 2.2.1 and 2.2.2, have the following form

$$u_1 = e^{\tau\rho}\left(a_1 + r_1 - u_r\right), \quad u_2 = e^{-\tau\overline{\rho}}(a_2 + r_2). \tag{2.3.1}$$

Now an easy computation shows that

$$\begin{aligned}\tau^{-1}Du_1\overline{u}_2 &= \left[e^{\tau\rho}\left(D\rho(a_1 + r_1 - u_r) + \tau^{-1}D(a_1 + r_1 - u_r)\right)\right]\\ &\qquad \times\left[e^{-\tau\rho}(\overline{a}_2 + \overline{r}_2)\right]\\ &= D\rho a_1\overline{a}_2 + M_1\end{aligned} \tag{2.3.2}$$

and

$$\begin{aligned}\tau^{-1}u_1\overline{Du_2} &= \left[e^{\tau\rho}(a_1 + r_1 - u_r)\right]\\ &\qquad \times\left[e^{-\tau\rho}\left(D\rho(\overline{a}_2 + \overline{r}_2) + \tau^{-1}\overline{D}(\overline{a}_2 + \overline{r}_2)\right)\right]\\ &= D\rho a_1\overline{a}_2 + M_2,\end{aligned} \tag{2.3.3}$$

where

$$\begin{aligned}M_1 = {} & D\rho r_1\overline{a}_2 + \tau^{-1}Da_1(\overline{a}_2 + \overline{r}_2) + \tau^{-1}Dr_1(\overline{a}_2 + \overline{r}_2) + D\rho(a_1 + r_1)\overline{r}_2\\ & - \tau^{-1}e^{-\tau\rho}Du_r(\overline{a}_2 + \overline{r}_2)\end{aligned}$$

and

$$\begin{aligned}M_2 = {} & D\rho(a_1 + r_1)\overline{r}_2 + \tau^{-1}a_1(\overline{Da_2} + \overline{Dr_2}) + D\rho r_1\overline{a}_2 + \tau^{-1}r_1(\overline{Da_2} + \overline{Dr_2})\\ & - e^{-\tau\rho}u_r D\rho(\overline{a}_2 + \overline{r}_2) + \tau^{-1}e^{-\tau\rho}u_r(\overline{Da_2} + \overline{Dr_2}).\end{aligned}$$

Now from (C.1.7), we obtain the following estimates

$$\left\|e^{-\tau\rho}u_r\right\|_{L^2(\Omega)} \leqslant C_1\tau^{-1}, \quad \left\|e^{-\tau\rho}Du_r\right\|_{L^2(\Omega)} \leqslant C_1, \tag{2.3.4}$$

and by a straightforward computation and a similar analysis as in the proof of Corollary 2.2.4, there exist two positive constants C_2 and τ_2 such that

$$\|M_j\|_{L^2(\Omega)} \leqslant C_2\tau^{-1}\left\|\overline{g}\right\|_{H^2(\Omega)}, \quad j = 1, 2, \tag{2.3.5}$$

hold true for all $\tau \geqslant \tau_2$. Thus, Corollary 2.2.4 and (2.3.5) imply that there exist two positive constants C_6 and τ_1 such that the estimate

$$
\begin{aligned}
&2\int_\Omega (A_1-A_2)\cdot D\rho a_1\overline{a_2}dx\\
&=\tau^{-1}\int_\Omega (A_1-A_2)\cdot(Du_1\overline{u}_2+u_1\overline{Du}_2)+(A_1^2-A_2^2+q_1-q_2)u_1\overline{u}_2\\
&\quad-\int_\Omega (A_1-A_2)\cdot(M_1+M_2)-\tau^{-1}\int_\Omega (A_2^2-A_1^2+q_2-q_1)u_1\overline{u_2}\\
&\leqslant \tau^{-1}\left|\langle(\Lambda_1-\Lambda_2)u_1,u_2\rangle_{L^2(\partial\Omega)}\right|+C_3\left\|M_1+M_2\right\|_{L^2(\Omega)}\\
&\quad+C_4\tau^{-1}\left\|e^{-\tau\varphi}u_1\right\|_{L^2(\Omega)}\left\|e^{\tau\varphi}u_2\right\|_{L^2(\Omega)}\\
&\leqslant C_5\left(e^{4\tau c}\left\|\Lambda_1^\sharp-\Lambda_2^\sharp\right\|+\tau^{-\frac12}+\tau^{-1}+\tau^{-1}\right)\left\|\overline{g}\right\|_{H^2(\Omega)}\\
&\leqslant C_6\left(e^{4\tau c}\left\|\Lambda_1^\sharp-\Lambda_2^\sharp\right\|+\tau^{-\frac12}\right)\left\|\overline{g}\right\|_{H^2(\Omega)},
\end{aligned}
\tag{2.3.6}
$$

holds true for all $\tau\geqslant\tau_1$. Hence, we have

$$
\begin{aligned}
&\left|(\xi+i\zeta)\cdot\int_\Omega (A_1-A_2)e^{\Phi_1+\overline{\Phi}_2}gdx\right|\\
&\qquad\leqslant C_6\left(e^{4\tau c}\left\|\Lambda_1^\sharp-\Lambda_2^\sharp\right\|+\tau^{-\frac12}\right)\left\|\overline{g}\right\|_{H^2(\Omega)}
\end{aligned}
\tag{2.3.7}
$$

for all $\tau\geqslant\tau_1$. Next, we use the last inequality to get information about the difference of A_1-A_2. To do that, we will use Lemma 2.3.1 in order to remove the function $e^{\Phi_1+\overline{\Phi}_2}$. Before to state the lemma we have to introduce new coordinates. Since every $x\in\mathbb{R}^n$ can be written as follows

$$
x=a\xi+b\zeta+x',\quad a=\xi\cdot x,\ b=\zeta\cdot x,
\tag{2.3.8}
$$

we consider the change of coordinates in $\mathbb{R}^n$ given by $x\mapsto(a,b,x')$.

Lemma 2.3.1. *Let $\xi,\zeta,\varsigma\in\mathbb{R}^n$ ($n\geqslant 3$) be orthogonal vectors such that $|\xi|=|\zeta|=1$. Consider the coordinates in $\mathbb{R}^n$ given by (2.3.8). If $W\in(L^\infty\cap\mathcal{E}')(\mathbb{R}^n;\mathbb{C}^n)$ and Φ satisfies*

$$(\xi+i\zeta)\cdot\nabla\Phi+(\xi+i\zeta)\cdot W=0$$

in $\mathbb{R}^n$ then

$$(\xi+i\zeta)\cdot\int_{\mathbb{R}^n}W(x)e^{i\varsigma\cdot x}e^{\Phi(x)}g(x)dx=(\xi+i\zeta)\cdot\int_{\mathbb{R}^n}W(x)e^{i\varsigma\cdot x}g(x)dx,$$

for all smooth function g depending only on x', that is $g(x)=g(x')$.

Remark 2.3.2. *The proof of this lemma for the case $g\equiv 1$ was given in [28]. See also Lemma* 2.6 *in [50]. The proof for any g depending only on x' is similar to the proof of Proposition* 3.3 *in [28]. For this reason, we omit the proof.*

Proposition 2.3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Let $\xi \in N \subset S^{n-1}$ and $\zeta \in S^{n-1}$ such that $\xi \cdot \zeta = 0$. Let $M > 0$ and $\gamma \in (0,1)$. Consider $A_1 \in \mathscr{A}(\Omega, M, \gamma)$, $A_2 \in \mathscr{A}(\Omega, M, 0)$ and $q_1, q_2 \in L^\infty(\Omega)$. If $A_1 = A_2$ on $\partial\Omega$ then there exist two positive constants τ_0 and $C > 0$ (both depending on n, Ω, M, γ) such that*

$$\left|\mu \cdot \int_\Omega (A_1 - A_2) g dx\right| \leqslant C\,|\mu| \left|\log \left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\|\right|^{-1/2} \|\overline{g}\|_{H^2(\Omega)} \tag{2.3.9}$$

holds true for all $\mu \in span\,\{\xi, \zeta\}$, provided that $\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\| \leqslant e^{-4c\tau_0}$.

Proof. We start by proving the proposition for the particular case $\mu = \xi + i\zeta$. The equations (C.1.1), (2.2.7) and (2.2.9) imply that

$$(\xi + i\zeta) \cdot \nabla(\Phi_1 + \overline{\Phi}_2) + i(\xi + i\zeta) \cdot [\chi_\Omega(A_1 - A_2)] = 0$$

in Ω. Notice that the above equation could be extended to all $\mathbb{R}^n$ by considering $A_1 - A_2 = 0$ on $\mathbb{R}^n \setminus \Omega$. Then applying Lemma 3.5.4 with $\varsigma = 0$, $W = i\chi_\Omega(A_1 - A_2)$, $\Phi = \Phi_1 + \overline{\Phi}_2$ and any function g depending only on x' (this kind of functions g satisfies (2.2.9)), we obtain

$$\begin{aligned}
&(\xi + i\zeta) \cdot \int_\Omega (A_1 - A_2) g e^{\Phi_1 + \overline{\Phi}_2} dx \\
&\quad = (\xi + i\zeta) \cdot \int_{\mathbb{R}^n} \chi_\Omega (A_1 - A_2) g e^{\Phi_1 + \overline{\Phi}_2} dx \\
&\quad = (\xi + i\zeta) \cdot \int_{\mathbb{R}^n} \chi_\Omega (A_1 - A_2) g dx \\
&\quad = (\xi + i\zeta) \cdot \int_\Omega (A_1 - A_2) g dx.
\end{aligned} \tag{2.3.10}$$

On the other hand, there exists $\tau_2 > 0$ such that

$$e^{-2\tau c} \leqslant \tau^{-1/2}, \tag{2.3.11}$$

for all $\tau \geqslant \tau_2$. Let $\tau_1 > 0$ be such that (2.3.7) is satisfied. Taking $\tau_0 = \max(\tau_1, \tau_2)$, it is easy to check that

$$\tau := \frac{1}{8} c^{-1} \left|\log \left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\|\right| \geqslant \tau_0,$$

whenever

$$\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\| \leqslant e^{-4c\tau_0}.$$

Thus, from (2.3.10) and replacing the above inequalities into (2.3.7), we get

$$\left|(\xi + i\zeta) \cdot \int_\Omega (A_1 - A_2) g dx\right| \leqslant C_1 \left|\log \left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\|\right|^{-1/2} \|\overline{g}\|_{H^2(\Omega)}. \tag{2.3.12}$$

By Remark 2.2.3, we can apply the previous arguments again, with $(\xi + i\zeta)$ replaced by $(\xi - i\zeta)$, to obtain

$$\left|(\xi - i\zeta) \cdot \int_\Omega (A_1 - A_2) g dx\right| \leqslant C_2 \left|\log \left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\|\right|^{-1/2} \|\overline{g}\|_{H^2(\Omega)}. \tag{2.3.13}$$

Hence, we conclude the proof by combining (2.3.12) and (2.3.13). □

In order to extract information about the magnetic potentials from the estimate (2.3.9), we shall rewrite its left-hand side as a Radon transform of a suitable function. In the next section, we briefly introduce the definition and some properties of the Radon transform.

2.3.1 Radon transform and its applications

Let f be a function on $\mathbb{R}^n$, integrable on each hyperplane in $\mathbb{R}^n$. These hyperplanes can be parametrized by its unit normal vector and distance to the origin, denoted by θ and s, respectively. We set

$$H(s,\theta) = \{x \in \mathbb{R}^n \ : \ \langle x, \theta \rangle = s\}$$

and in this setting, the Radon transform of f is defined by

$$(\boldsymbol{R}f)(s,\theta) = \int_H f(x) d\mu_H = \int\limits_{\theta^\perp} f(s\theta + y) dy,$$

whenever the integral exists. Here $\theta^\perp$ denotes the set of orthogonal vectors to θ. This is the definition of the Radon transform with respect to the origin, but later on will need to know this transform at some arbitrary point in $\mathbb{R}^n$. In this case, the natural definition is as follows. For $y_0 \in \mathbb{R}^n$, we set

$$H_{y_0} = \{x \in \mathbb{R}^n \ : \ \langle x - y_0, \theta \rangle = s\}$$

for some $\theta \in S^{n-1}$ and $s \in \mathbb{R}$. With respect to these parameters we define

$$\boldsymbol{R}_{y_0} f(s,\theta) = \int_{H_{y_0}} f d\mu_{H_{y_0}},$$

where $\mu_{H_{y_0}}$ denotes the natural measure on the hyperplane H_{y_0}. It is easy to check that the following relation

$$\boldsymbol{R}_{y_0} f(s,\theta) = (\boldsymbol{R}f)(s + \langle y_0, \theta \rangle, \theta), \tag{2.3.14}$$

holds true for all $y_0 \in \mathbb{R}^n$, $\theta \in S^{n-1}$ and $s \in \mathbb{R}$. Now we move to define the Fourier transform with respect to the first variable of a function $F : \mathbb{R} \times S^{n-1} \to \mathbb{R}$ as follows

$$\widehat{F}(\sigma,\theta) = (2\pi)^{-1/2} \int\limits_{\mathbb{R}} e^{-is\sigma} F(s,\theta) ds.$$

For $\alpha \geqslant 0$, we also define the Sobolev space $H^\alpha(\mathbb{R} \times S^{n-1})$ as the subspace of $L^2(\mathbb{R} \times S^{n-1})$ with the norm

$$\|F\|_{H^\alpha(\mathbb{R}\times S^{n-1})} = \left(\int\limits_{S^{n-1}} \int\limits_{\mathbb{R}} (1+\sigma^2)^\alpha \left| \widehat{F}(\sigma,\theta) \right|^2 d\sigma d\theta \right)^{1/2}.$$

The following two results can be found in [33]: for each $\alpha \geqslant 0$ there exist two positive constants C_1 and C_2 (both depending on α and n) such that

$$C_1 \|f\|_{H^\alpha(\mathbb{R}^n)} \leqslant \|\boldsymbol{R}f\|_{H^{\alpha+(n-1)/2}(\mathbb{R}\times S^{n-1})} \leqslant C_2 \|f\|_{H^\alpha(\mathbb{R}^n)}, \tag{2.3.15}$$

whenever f has a compact support. Moreover, for all $f \in H^1(\mathbb{R}^n)$ with compact support, the following identity holds in the sense of the distributions in $C_0^\infty(\mathbb{R})$ and for all $\theta \in S^{n-1}$

$$\theta_i \frac{\partial}{\partial s}(\boldsymbol{R}f)(\cdot,\theta) = \boldsymbol{R}(\partial_{x_i} f)(\cdot,\theta), \quad i = 1, 2, \ldots, n, \tag{2.3.16}$$

where θ_i denotes the i-th coordinate of θ. This identity give us a relation between the natural partial derivatives of a function and the scalar derivative of its Radon transform. This fact will be useful in our approach. From (2.3.15), we deduce that the Radon transform is injective in $H^\alpha(\mathbb{R}^n)$. This is still true when we only consider the Radon transform on $\mathbb{R}\times M$, where M is an open set of S^{n-1}. A quantitative estimate of the aforementioned result was obtained by Caro *et al.*, see Theorem 2.5 in [10]. Before stating their result, we introduce the set X as the subspace of $L^1(\mathbb{R}^n)$ with the norm

$$\|F\|_X = \int_{\mathbb{R}} (1+|s|)^n \|\boldsymbol{R}F(s,\cdot)\|_{L^1(S^{n-1})}\, ds.$$

and recall the distance on the sphere: $d_{S^{n-1}}(x,y) = \arccos(\langle x,y\rangle)$.

Theorem 2.3.4. *Let $M \geqslant 1, \alpha > 0$ and $\beta \in (0,1)$. Given $y_0 \in \mathbb{R}^n$ and $\theta_0 \in S^{n-1}$, consider the set*

$$\Gamma = \left\{\theta \in S^{n-1} \ : \ d_{S^{n-1}}(\theta_0,\theta) < \arcsin\beta\right\}$$

and the domain of dependence of the Radon transform by

$$E = \{x \in \mathbb{R}^n \ : \ \langle \theta, x - y_0\rangle = s \ , \ s \in (-\alpha,\alpha) \ , \ \theta \in \Gamma\}.$$

Assume that there exist two constants p, with $1 \leqslant p < \infty$ and λ, with $0 < \lambda < p^{-1}$; such that a function F satisfies the following conditions:

(a). $\chi_E F \in X \cap L^\infty(\mathbb{R}^n)$, where χ_E denotes the characteristic function of the set E. Moreover

$$\|F\|_{L^\infty(E)} + \|\chi_E F\|_X \leqslant M.$$

(b). $y_0 \in \operatorname{supp} F$ and $\operatorname{supp} F \subset \{x \in \mathbb{R}^n \ : \ \langle x - y_0, \theta_0\rangle \leqslant 0\}$.

(c). The function F satisfies the following (λ, p)-Besov regularity

$$\int_{\mathbb{R}^n} \frac{\|\chi_E F(\cdot) - (\chi_E F)(\cdot - y)\|^p_{L^p(\mathbb{R}^n)}}{|y|^{n+\lambda p}}\, dy \leqslant M^p.$$

Then there exists a positive constant C (depending on $G, M, \alpha, \beta, \lambda$), such that

$$\|F\|_{L^p(G)} \leqslant C \left|\log \int_{-\alpha}^{\alpha} (1+|s|)^n \|\boldsymbol{R}_{y_0} F(s,\cdot)\|_{L^1(\Gamma)}\, ds\right|^{-\lambda/2},$$

where

$$G = \left\{ x \in \mathbb{R}^n \ : \ |x - y_0| < \frac{\alpha}{8\cosh(8\pi/\beta)} \right\}. \tag{2.3.17}$$

Remark 2.3.5. *In our context, the constant β stands for the size of the set $N \subset S^{n-1}$. Recall that N is an open subset of S^{n-1} from which are defined the sets F_N and B_N, neighborhoods of the illuminated and shadowed face of the boundary, respectively. See (1.1.2). The interval $(-\alpha, \alpha)$ is where we have control of the Radon transform $\boldsymbol{R}F(\cdot, \theta)$, with $\theta \in S^{n-1}$ and $F \in X$. Notice that for fixed $y_0 \in \mathbb{R}^n$ and $\beta > 0$, we can take α large enough so that $\Omega \subset G$. We will use these facts in the proof of Theorem 1.2.4.*

2.3.2 Proof of Theorem 1.1.4

We start by rewriting the estimate from Proposition 2.3.3 in the natural coordinates of the Radon transform of $\chi_\Omega(A_1 - A_2)$. More precisely

Corollary 2.3.6. *If we consider the open set in S^{n-1}*

$$M = \bigcup_{\xi \in N} [\xi]^{\perp}, \tag{2.3.18}$$

then for any $\widetilde{g} \in C^\infty(\mathbb{R})$ there exist two positive constants C and τ_0 (both depending on n, Ω and the a priori bounds of $\|A_1\|_{C^{2,\gamma}(\overline{\Omega})}$, $\|A_2\|_{C^2(\overline{\Omega})}$ and $\|q_j\|_{L^\infty(\Omega)}$, $j = 1, 2$) such that the following estimate

$$\begin{aligned} &\left| \mu \cdot \int_{\mathbb{R}} \widetilde{g}(s) (\boldsymbol{R}\, [\chi_\Omega (A_1 - A_2)])(s, \theta) ds \right| \\ &\qquad \leqslant C\, |\mu| \left| \log \left\| \Lambda_1^\sharp - \Lambda_2^\sharp \right\| \right|^{-1/2} \|\widetilde{g}\|_{H^2(\mathbb{R})}, \end{aligned} \tag{2.3.19}$$

holds true for all $\theta \in M$ and for all $\mu \in \theta^\perp$.

Proof. The main idea of the proof is to see the left-hand side of (2.3.9) as the Radon transform of a suitable function. We start by considering $\xi \in N \subset S^{n-1}$ and $\zeta \in S^{n-1}$ such that $\xi \cdot \zeta = 0$ and we take some $\theta \in [\xi, \zeta]^\perp$ with $|\theta| = 1$. Thus, every $x \in \mathbb{R}^n$ can be written as

$$x = t\xi + r\zeta + s\theta + x', \quad x' \in [\xi, \zeta, \theta]^\perp.$$

This decomposition can be done since $n \geqslant 3$. Now we consider the change of coordinates in $\mathbb{R}^n$ defined by $\Psi : x \mapsto (t, r, s, x')$; and a straightforward computation shows that if $g \in C^\infty(\mathbb{R}^n)$ satisfies $(\xi + i\zeta) \cdot \nabla g = 0$, then the function $\widetilde{g} := \overline{g} \circ \Psi^{-1}$ satisfies

$$(\partial_t - i\partial_r)\widetilde{g} = 0, \tag{2.3.20}$$

where ∂_t and ∂_r denote the partial derivative with respect to t and r, respectively. Notice that any function $\widetilde{g} := \widetilde{g}(s)$ that depends only on the variable s, satisfies (2.3.20). For

Ψ-coordinates we have $dx = dx'dtdrds$ and for every $\mu \in [\xi, \zeta]$, we obtain

$$\begin{aligned}
&\mu \cdot \int_{\Omega} (A_1 - A_2) g dx = \mu \cdot \int_{\mathbb{R}^n} \chi_\Omega (A_1 - A_2) g dx \\
&= \mu \cdot \int_{\mathbb{R}^3} \int_{[\xi,\zeta,\theta]^\perp} \left[\chi_\Omega (A_1 - A_2) \circ \Psi^{-1}\right] \left[g \circ \Psi^{-1} dx'\right] dtdrds \\
&= \mu \cdot \int_{\mathbb{R}} \widetilde{g}(s) \left(\int_{\mathbb{R}^2} \int_{[\xi,\zeta,\theta]^\perp} \left[\chi_\Omega (A_1 - A_2)\right] (t\xi + r\zeta + s\theta + x') dx' dtdr \right) ds \\
&= \mu \cdot \int_{\mathbb{R}} \widetilde{g}(s) \left(\int_{\theta^\perp} \left[\chi_\Omega (A_1 - A_2)\right] (s\theta + y) dy \right) ds \\
&= \mu \cdot \int_{\mathbb{R}} \widetilde{g}(s) (\boldsymbol{R} \left[\chi_\Omega (A_1 - A_2)\right])(s, \theta) ds.
\end{aligned}$$

This equality and estimate (2.3.9) imply (2.3.19). $\square$

In particular, the estimate (2.3.19) holds for the vectors $\mu_{ij} = \theta_i e_j - \theta_j e_i$ with $i, j = 1, 2, \ldots, n$. Here $(e_i)_{i=1}^n$ denotes the canonical basis of $\mathbb{R}^n$ and θ_i the i-th component of θ. Denoting $\widetilde{A} = \chi_\Omega (A_1 - A_2)$ and since $A_1 = A_2$ on $\partial\Omega$, it follows that $\widetilde{A}$ belongs to $H^1(\mathbb{R}^n)$ and also has compact support. Thus, from (2.3.16) we get the following identity

$$\begin{aligned}
&\mu_{i,j} \cdot \int_{\mathbb{R}} \frac{\partial}{\partial s} \widetilde{h}(s) (\boldsymbol{R} \left[\chi_\Omega (A_1 - A_2)\right])(s, \theta) ds \\
&= \int_{\mathbb{R}} \frac{\partial}{\partial s} \widetilde{h}(s) \left[\theta_i e_j - \theta_j e_i\right] \cdot (\boldsymbol{R} \widetilde{A})(s, \theta) ds \\
&= \int_{\mathbb{R}} \frac{\partial}{\partial s} \widetilde{h}(s) \left[\theta_i \left(\boldsymbol{R} \widetilde{A}_j\right)(s, \theta) - \theta_j \left(\boldsymbol{R} \widetilde{A}_i\right)(s, \theta)\right] ds \\
&= -\int_{\mathbb{R}} \widetilde{h}(s) \left[\theta_i \frac{\partial}{\partial s} \left(\boldsymbol{R} \widetilde{A}_j\right)(s, \theta) - \theta_j \frac{\partial}{\partial s} \left(\boldsymbol{R} \widetilde{A}_i\right)(s, \theta)\right] ds \\
&= -\int_{\mathbb{R}} \widetilde{h}(s) \left[\boldsymbol{R} \left(\partial_{x_i} \widetilde{A}_j - \partial_{x_j} \widetilde{A}_i\right)\right](s, \theta) ds,
\end{aligned}$$

for all $\widetilde{h} \in C_0^\infty(\mathbb{R})$ and all $i, j = 1, 2, \ldots, n$. From this and (2.3.19) it follows that for all $\theta \in M$ we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \widetilde{h}(s) \left[\boldsymbol{R} \left(\partial_{x_i} \widetilde{A}_j - \partial_{x_j} \widetilde{A}_i\right)\right](s, \theta) ds \right| \\
&\quad = \left| \mu_{i,j} \cdot \int_{\mathbb{R}} \frac{\partial}{\partial s} \widetilde{h}(s) (\boldsymbol{R} \left[\chi_\Omega (A_1 - A_2)\right])(s, \theta) ds \right| \\
&\quad \leqslant C \left|\mu_{i,j}\right| \left| \log \left\| \Lambda_1^\sharp - \Lambda_2^\sharp \right\| \right|^{-1/2} \left\| \partial_s \widetilde{h} \right\|_{H^2(\mathbb{R})} \\
&\quad \leqslant C \left| \log \left\| \Lambda_1^\sharp - \Lambda_2^\sharp \right\| \right|^{-1/2} \left\| \widetilde{h} \right\|_{H^3(\mathbb{R})},
\end{aligned}$$

which implies that

$$\left\| \boldsymbol{R}\left(\partial_{x_i}\widetilde{A}_j - \partial_{x_j}\widetilde{A}_i\right)\right\|_{H^{-3}(\mathbb{R};L^\infty(M))} \leqslant C\left|\log\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\|\right|^{-1/2}.$$

On the other hand, applying (2.3.15) with $\alpha = 0$, we obtain

$$\left\| \boldsymbol{R}\left(\partial_{x_i}\widetilde{A}_j - \partial_{x_j}\widetilde{A}_i\right)\right\|_{H^{\frac{n-1}{2}}(\mathbb{R};L^2(M))} \leqslant C_1\left\|\partial_{x_i}\widetilde{A}_j - \partial_{x_j}\widetilde{A}_i\right\|_{L^2(\mathbb{R}^n)} \leqslant C_2.$$

Thus, by a standard interpolation between the spaces $H^{-3}(\mathbb{R};L^\infty(M))$ and $H^{\frac{n-1}{2}}(\mathbb{R};L^2(M))$, we have

$$\begin{aligned}\left\| \boldsymbol{R}\left(\partial_{x_i}\widetilde{A}_j - \partial_{x_j}\widetilde{A}_i\right)\right\|_{L^2(\mathbb{R};L^{(n+5)/3}(M))} \\ \leqslant C_3\left|\log\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\|\right|^{-\frac{1}{2}(n-1)/(n+5)}.\end{aligned} \tag{2.3.21}$$

The next step will be to verify the three conditions of Theorem 2.3.4 for the functions $F_{i,j} := \partial_{x_i}\widetilde{A}_j - \partial_{x_j}\widetilde{A}_i$, for fixed $i \neq j$; $i,j \in \{1,2,\dots,n\}$. Let us start with the supporting condition (b). Indeed, take $\theta_0 \in M$ and by translation, there exists $y_0 \in \operatorname{supp} F_{i,j}$ such that

$$\operatorname{supp} F_{i,j} \subset \{x \in \mathbb{R}^n \ : \ \langle x - y_0, \theta_0\rangle \leqslant 0\}.$$

This can be done because Ω is a bounded open set. Since M is an open neighborhood of θ_0 and from estimate (2.3.21), we can control the Radon transform of $F_{i,j}$ for all $s \in \mathbb{R}$ and for all $\theta \in M$. Thus, from Remark 2.3.5, there exists $\beta \in (0,1)$ such that the condition (a) is satisfied for any $\alpha > 0$. Moreover, by taking α large enough it follows that $\operatorname{supp} F_{i,j} \subset \overline{\Omega} \subset G$, where G is defined by (2.3.17). The condition (c) is satisfied for $p = 2$ and $0 < \lambda < 1/2$. Thus, Theorem 2.3.4 ensures that there exists $C > 0$ such that

$$\|F_{i,j}\|_{L^2(\mathbb{R}^n)} \leqslant C\left|\log\int_{-\alpha}^{\alpha}(1+|s|)^n\left\|\boldsymbol{R}_{y_0}F_{i,j}(s,\cdot)\right\|_{L^1(\Gamma)}ds\right|^{-\lambda/2}. \tag{2.3.22}$$

Here the set Γ is where we have the control of the Radon transform on the θ-variable. In our case, see the estimate (2.3.21), we have the control on M. Now we set

$$L = \sup_{\theta\in M}\|(1+|\cdot - \langle\theta,y_0\rangle|)^n\|_{L^2(|s|\leqslant\alpha+|y_0|)}$$

and denote by $|M|$ the measure of M. Then, the inequalities (2.3.21)-(2.3.22), Fubini's theorem and Hölder's inequality applied twice, imply that

$$
\begin{aligned}
&\int_{-\alpha}^{\alpha} (1+|s|)^n \left\| \boldsymbol{R}_{y_0} F_{i,j}(s,\cdot) \right\|_{L^1(M)} ds \\
&= \int_{-\alpha}^{\alpha} (1+|s|)^n \int_M \left| (\boldsymbol{R} F_{i,j})(s + \langle \theta, y_0 \rangle, \theta) \right| d\theta ds \\
&\leqslant \int_M \int_{-(\alpha+|y_0|)}^{\alpha+|y_0|} (1 + |s - \langle \theta, y_0\rangle|)^n \left| (\boldsymbol{R} F_{i,j})(s,\theta) \right| ds d\theta \\
&\leqslant \int_M \left\| (1 + |\cdot - \langle \theta, y_0 \rangle|)^n \right\|_{L^2(|s| \leqslant \alpha+|y_0|)} \left\| (\boldsymbol{R} F_{i,j})(\cdot,\theta) \right\|_{L^2(|s|\leqslant \alpha+|y_0|)} d\theta \\
&\leqslant L \int_M \left(\int_{\mathbb{R}} \left| \boldsymbol{R} F_{i,j}(s,\theta) \right|^2 ds \right)^{1/2} d\theta \\
&\leqslant L\, |M|^{\frac{n+2}{n+5}} \left(\int_M \left(\int_{\mathbb{R}} \left| \left(\boldsymbol{R} F_{i,j} \right)(s,\theta) \right|^2 \right)^{(n+5)/6} d\theta \right)^{3/(n+5)} \\
&= L\, |M|^{\frac{n+2}{n+5}} \left\| \boldsymbol{R} \left(\partial_{x_i} \widetilde{A}_j - \partial_{x_j} \widetilde{A}_i \right) \right\|_{L^2(\mathbb{R}; L^{(n+5)/3}(M))} \\
&\leqslant C_4 \left| \log \left\| \Lambda_1^\sharp - \Lambda_2^\sharp \right\| \right|^{-\frac{1}{2}(n-1)/(n+5)} .
\end{aligned}
$$

We conclude the proof by taking logarithm to both sides of the above inequality and taking into account the estimate (2.3.22).

2.4 Proof of log-type estimate for the electric potentials

The goal of this section is to prove Theorem 1.1.5. This will be done by combining the gauge invariance for the DN map, the stability result already proved for the magnetic fields and a Hodge decomposition derived by Tzou [50]. We recall this decomposition in the following lemma.

Lemma 2.4.1. *Let $\Omega \subset \mathbb{R}^n$ be a simply-connected open bounded set with smooth boundary. If $A_1, A_2 \in W^{2,p}(\Omega)$ with $p \geqslant 2$, and $A_1 = A_2$ on $\partial\Omega$. Then there exist a constant $C > 0$ and $\omega \in W^{3,p}(\Omega) \cap H_0^1(\Omega)$ such that*

$$
\left\| A_1 - A_2 - d\omega \right\|_{W^{1,p}(\Omega)} \leqslant C \left\| d(A_1 - A_2) \right\|_{L^p(\Omega)}
$$

and

$$
\left\| \omega \right\|_{W^{3,p}(\Omega)} \leqslant C \left\| A_1 - A_2 \right\|_{W^{2,p}(\Omega)} .
$$

From now on we consider the bounded open set Ω to be simply-connected with connected smooth boundary. Let $A_1, A_2 \in W^{2,\infty}(\Omega)$ be two magnetic potentials and $q_1, q_2 \in L^\infty(\Omega)$ be two electric potentials. Also fix $p \in \mathbb{N}$ with $p > n$. Then, by Morrey's inequality and Lemma 2.4.1, there exist a constant $C > 0$ and $w \in W^{3,p}(\Omega) \cap H_0^1(\Omega)$ such that

$$
\left\| A_1 - A_2 - \nabla\omega \right\|_{C^{0,1-\frac{n}{p}}(\overline{\Omega})} \leqslant C \left\| d(A_1 - A_2) \right\|_{L^p(\Omega)} \tag{2.4.1}
$$

and

$$\|\omega\|_{L^\infty(\Omega)} + \|\nabla\omega\|_{L^\infty(\Omega)} + \|\Delta\omega\|_{L^\infty(\Omega)} \leqslant C\, \|A_1 - A_2\|_{W^{2,p}(\Omega)}\, . \tag{2.4.2}$$

Now by setting $\widetilde{A}_1 = A_1 - \nabla\omega/2$ and $\widetilde{A}_2 = A_2 + \nabla\omega/2$ and by an easy computation, see for instance Lemma 3.1 in [28], we have the identities

$$e^{i\omega/2}\mathcal{L}_{\widetilde{A}_1,q_1}e^{-i\omega/2} = \mathcal{L}_{\widetilde{A_1},q_1}\ ,\ \Lambda_{A_1,q_1} = \Lambda_{\widetilde{A}_1,q_1} \tag{2.4.3}$$

and

$$e^{-i\omega/2}\mathcal{L}_{\widetilde{A}_1,q_1}e^{i\omega/2} = \mathcal{L}_{\widetilde{A_2},\overline{q}_2}\ ,\ \Lambda_{A_2,\overline{q}_2} = \Lambda_{\widetilde{A}_2,\overline{q}_2}. \tag{2.4.4}$$

To derive stability estimates for the magnetic potentials we have used the integral identity (2.1.2) to isolate $A_1 - A_2$ and then using special solutions for the magnetic Schrödinger equation, we obtained the estimate from Corollary 2.2.4. We follow similar ideas to obtain an estimate for $q_1 - q_2$. We denote $\widetilde{\Lambda}_i = \Lambda_{\widetilde{A}_i,q_i}$ for $i = 1, 2$. Now let $U_1, U_2 \in H^1(\Omega)$ function belong to $H^1(\Omega)$ such that $\mathcal{L}_{\widetilde{A}_1,q_1}U_1 = 0$ and $\mathcal{L}_{\widetilde{A}_2,\overline{q_2}}U_2 = 0$. Then, by Lemma 2.1.2, we deduce the identity

$$\begin{aligned}&\left\langle(\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2)U_1, U_2\right\rangle_{L^2(\partial\Omega)}\\ &= \int_\Omega \left[(\widetilde{A}_1 - \widetilde{A}_2)\cdot(DU_1\overline{U}_2 + U_1\overline{DU}_2) + (\widetilde{A}_1^2 - \widetilde{A}_2^2 + q_1 - q_2)U_1\overline{U}_2\right].\end{aligned} \tag{2.4.5}$$

The analogous of Proposition 2.1.6, with Λ_i replaced by $\widetilde{\Lambda}_i$ with $i = 1, 2$; is given in the following Proposition.

Proposition 2.4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Consider three positive constants M, $\sigma \in (0, 1/2)$ and $\gamma \in (0, 1)$. Let $A_1 \in \mathscr{A}(\Omega, M, \gamma)$, $A_2 \in \mathscr{A}(\Omega, M, 0)$ with $A_1 = A_2$ on $\partial\Omega$; and $q_1, q_2 \in \mathscr{Q}(\Omega, M, \sigma)$. If $U_1, U_2 \in H^1(\Omega)$ satisfy $\mathcal{L}_{\widetilde{A}_1,q_1}U_1 = 0$ and $\mathcal{L}_{\widetilde{A}_2,\overline{q}_2}U_2 = 0$, then there exist two positive constants τ_0 and C (both depending on $n, \Omega, M, \sigma, \gamma$) such that the estimate*

$$\begin{aligned}&\left|\left\langle(\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2)U_1, U_2\right\rangle_{L^2(\partial\Omega)}\right|\\ &\leqslant C\left\|\widetilde{\Lambda}_1^\sharp - \widetilde{\Lambda}_2^\sharp\right\|\left(\|U_1\|_{H^1(\Omega)}\|U_2\|_{H^1(\Omega)} + e^{\tau c}\|U_1\|_{H^1(\Omega)}\left\|e^{\tau\xi\cdot x}U_2\right\|_{L^2(\partial\Omega)}\right)\\ &\quad + C\tau^{-\frac{1}{2}}\left\|e^{-\tau\xi\cdot x}(\mathcal{L}_{\widetilde{A}_1,q_1} - \mathcal{L}_{\widetilde{A}_2,q_2})U_1\right\|_{L^2(\Omega)}\left\|e^{\tau\xi\cdot x}U_2\right\|_{L^2(\partial\Omega)}\\ &\quad + C\left\|\widetilde{A}_1 - \widetilde{A}_2\right\|_{L^\infty(\Omega)}\left\|e^{-\tau\xi\cdot x}U_1\right\|_{L^2(\partial\Omega)}\left\|e^{\tau\xi\cdot x}U_2\right\|_{L^2(\partial\Omega)}\end{aligned} \tag{2.4.6}$$

holds true for all $\tau \geqslant \tau_0$ and for all $\xi \in N$.

Remark 2.4.3. *Here the sets $\mathscr{A}(\Omega, M, \gamma)$, $\mathscr{A}(\Omega, M, 0)$ and $\mathscr{Q}(\Omega, M, \sigma)$ denote the class of admissible magnetic and electric potentials defined in definitions 1.1.2 and 1.1.3. Also we denote $\widetilde{\Lambda}_i = \widetilde{\Lambda}_i^\sharp$ with $i = 1, 2$. Finally, recall that N denote an open set of S^{n-1} as in the statement of Theorem 1.2.4.*

Proof. The proof is similar to the proof of Proposition 2.1.6, with A_i replaced by $\widetilde{A}_i$ for $i = 1, 2$. We give the proof only for completeness and we will take extra care when the term $\widetilde{A}_1 - \widetilde{A}_2 = A_1 - A_2 - \omega$ appears in the following computations. Throughout this proof, we take into account the notation from Proposition 2.1.6. Let us begin with the following identity

$$\begin{aligned}\left\langle (\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2) U_1, U_2 \right\rangle_{L^2(\partial\Omega)} &= \left\langle \chi(\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2) U_1, U_2 \right\rangle_{L^2(\partial\Omega)} \\ &\quad + \left\langle (1-\chi)(\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2) U_1, U_2 \right\rangle_{L^2(\partial\Omega)}.\end{aligned} \tag{2.4.7}$$

We estimate the first term of the right-hand side in the above identity as follows

$$\left| \int_{\partial\Omega} \chi(\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2) U_1 \overline{U}_2 dS \right| \leqslant \left\| \Lambda_1^\sharp - \Lambda_2^\sharp \right\| \|U_1\|_{H^1(\Omega)} \|U_2\|_{H^1(\Omega)}, \tag{2.4.8}$$

whenever $U_1 \in H^1(\Omega)$ satisfies the vanishing condition on E. For the second term we will use the Carleman estimate given by Proposition 2.1.4. Recall that we denoted by N an open subset of S^{n-1} as in the statement of Theorem 1.2.4. Since χ is equal to 1 on $\Omega_{-,\varepsilon}(\xi)$, we get the following estimate for every $\xi \in N$

$$\begin{aligned}&\left| \int_{\partial\Omega} (1-\chi)(\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2) U_1 \overline{U}_2 dS \right| \\ &= \left| \int_{\Omega_{-,\varepsilon}(\xi) \cup (\partial\Omega \setminus \Omega_{-,\varepsilon}(\xi))} (1-\chi)(\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2) U_1 \overline{U}_2 dS \right| \\ &= \left| \int_{\partial\Omega \setminus \Omega_{-,\varepsilon}(\xi)} (1-\chi)(\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2) U_1 \overline{U}_2 dS \right| \\ &\leqslant C_1 \left\| e^{-\tau\xi\cdot x} (\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2) U_1 \right\|_{L^2(\partial\Omega \setminus \Omega_{-,\varepsilon}(\xi))} \left\| e^{\tau\xi\cdot x} U_2 \right\|_{L^2(\partial\Omega \setminus \Omega_{-,\varepsilon}(\xi))}.\end{aligned} \tag{2.4.9}$$

We now estimate the $L^2(\partial\Omega \setminus \Omega_{-,\varepsilon}(\xi))$-norm in the above inequality. Let us introduce an auxiliary function w_1 satisfying

$$\begin{cases} \mathcal{L}_{\widetilde{A}_2, q_2} w_1 = 0, \\ w_1|_{\partial\Omega} = U_1|_{\partial\Omega}. \end{cases} \tag{2.4.10}$$

Now, since $U_1 \in H^1(\Omega)$ and $\mathcal{L}_{\widetilde{A}_2, q_2}(w_1 - U_1) = (\mathcal{L}_{\widetilde{A}_1, q_1} - \mathcal{L}_{\widetilde{A}_2, q_2}) U_1$, it follows that $\mathcal{L}_{A_2, q_2}(w - u_1) \in L^2(\Omega)$. Moreover, since w_1 satisfies (2.4.10), we have $w_1 - U_1 \in H_0^1(\Omega)$.

Hence, the Carleman estimate (2.1.9) and Remark 2.1.5, imply that

$$
\begin{aligned}
&\left\| e^{-\tau\xi\cdot x}(\widetilde{\Lambda}_1-\widetilde{\Lambda}_2)U_1\right\|_{L^2(\partial\Omega\setminus\Omega_{-,\varepsilon}(\xi))} \\
&= \left\| e^{-\tau\xi\cdot x}\left(\partial_\nu(U_1-w_1)+i\nu\cdot(\widetilde{A}_1-\widetilde{A}_2)U_1\right)\right\|_{L^2(\partial\Omega\setminus\Omega_{-,\varepsilon}(\xi))} \\
&\leqslant \left\| e^{-\tau\xi\cdot x}\partial_\nu(U_1-w_1)\right\|_{L^2(\partial\Omega\setminus\Omega_{-,\varepsilon}(\xi))} \\
&\qquad + \left\|\widetilde{A}_1-\widetilde{A}_2\right\|_{L^\infty(\Omega)}\left\| e^{-\tau\xi\cdot x}U_1\right\|_{L^2(\partial\Omega\setminus\Omega_{-,\varepsilon}(\xi))} \\
&\leqslant \frac{1}{\sqrt{\varepsilon}}\left\|\sqrt{\langle\xi\cdot\nu(\cdot)\rangle}e^{-\tau\xi\cdot x}\partial_\nu(U_1-w_1)\right\|_{L^2(\partial\Omega\setminus\Omega_{-,\varepsilon}(\xi))} \\
&\qquad + \left\|\widetilde{A}_1-\widetilde{A}_2\right\|_{L^\infty(\Omega)}\left\| e^{-\tau\xi\cdot x}U_1\right\|_{L^2(\partial\Omega\setminus\Omega_{-,\varepsilon}(\xi))} \\
&\leqslant \frac{1}{\sqrt{\varepsilon}}\left\|\sqrt{\langle\xi\cdot\nu(\cdot)\rangle}e^{-\tau\xi\cdot x}\partial_\nu(U_1-w_1)\right\|_{L^2(\Omega_{+,0}(\xi))} \\
&\qquad + \left\|\widetilde{A}_1-\widetilde{A}_2\right\|_{L^\infty(\Omega)}\left\| e^{-\tau\xi\cdot x}U_1\right\|_{L^2(\partial\Omega)} \\
&\leqslant \frac{C_2}{\sqrt{\varepsilon}}\left(\left\| e^{-\tau\xi\cdot x}\partial_\nu(U_1-w_1)\right\|_{L^2(\partial\Omega_{-,0}(\xi))}\right. \\
&\qquad\qquad \left. +\tau^{-\frac{1}{2}}\left\| e^{-\tau\xi\cdot x}(\mathcal{L}_{\widetilde{A}_1,q_1}-\mathcal{L}_{\widetilde{A}_2,q_2})U_1\right\|_{L^2(\Omega)}\right) \\
&\qquad\qquad\qquad + \left\|\widetilde{A}_1-\widetilde{A}_2\right\|_{L^\infty(\Omega)}\left\| e^{-\tau\xi\cdot x}U_1\right\|_{L^2(\partial\Omega)}.
\end{aligned}
\tag{2.4.11}
$$

Now we estimate the $L^2(\partial\Omega_{-,0}(\xi))$-norm in the last inequality as follows

$$
\begin{aligned}
&\left\| e^{-\tau\xi\cdot x}\partial_\nu(U_1-w_1)\right\|_{L^2(\partial\Omega_{-,0}(\xi))} \\
&= \left\| e^{-\tau\xi\cdot x}\left[(\widetilde{\Lambda}_1-\widetilde{\Lambda}_2)U_1-i\nu\cdot(\widetilde{A}_1-\widetilde{A}_2)U_1\right]\right\|_{L^2(\partial\Omega_{-,0}(\xi))} \\
&\leqslant \left\| e^{-\tau\xi\cdot x}(\widetilde{\Lambda}_1-\widetilde{\Lambda}_2)U_1\right\|_{L^2(\partial\Omega_{-,0}(\xi))} \\
&\qquad + \left\| e^{-\tau\xi\cdot x}i\nu\cdot(\widetilde{A}_1-\widetilde{A}_2)U_1\right\|_{L^2(\partial\Omega_{-,0}(\xi))} \\
&= \left\| e^{-\tau\xi\cdot x}\chi(\widetilde{\Lambda}_1-\widetilde{\Lambda}_2)U_1\right\|_{L^2(\partial\Omega)} \\
&\qquad + \left\| e^{-\tau\xi\cdot x}i\nu\cdot(\widetilde{A}_1-\widetilde{A}_2)U_1\right\|_{L^2(\partial\Omega_{-,0}(\xi))} \\
&\leqslant e^{\tau c}\left\|\Lambda_1^\sharp-\Lambda_2^\sharp\right\|\left\|U_1\right\|_{H^{\frac{1}{2}}(\partial\Omega)}+\left\|\widetilde{A}_1-\widetilde{A}_2\right\|_{L^\infty(\Omega)}\left\| e^{-\tau\xi\cdot x}U_1\right\|_{L^2(\partial\Omega)}.
\end{aligned}
\tag{2.4.12}
$$

Thus, replacing (2.4.11) and (2.4.12) into (2.4.9) gives us

$$
\begin{aligned}
&\left|\int_{\partial\Omega}(1-\chi)(\widetilde{\Lambda}_1-\widetilde{\Lambda}_2)U_1\overline{U}_2 dS\right| \\
&\leqslant C_4\left(\varepsilon^{-1/2}e^{\tau c}\left\|\widetilde{\Lambda}_1^\sharp-\widetilde{\Lambda}_2^\sharp\right\|\|U_1\|_{H^1(\Omega)}\right. \\
&\qquad +\varepsilon^{-1/2}\tau^{-\frac{1}{2}}\left\|e^{-\tau\xi\cdot x}(\mathcal{L}_{\widetilde{A}_1,q_1}-\mathcal{L}_{\widetilde{A}_2,q_2})U_1\right\|_{L^2(\Omega)} \\
&\qquad \left.+\left\|\widetilde{A}_1-\widetilde{A}_2\right\|_{L^\infty(\Omega)}\left\|e^{-\tau\xi\cdot x}U_1\right\|_{L^2(\partial\Omega)}\right)\left\|e^{\tau\xi\cdot x}U_2\right\|_{L^2(\partial\Omega)}.
\end{aligned}
\tag{2.4.13}
$$

We conclude the proof by replacing (2.4.8) and (2.4.13) into (2.4.7).

□

Corollary 2.4.4. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set with smooth boundary. Consider three positive constants M, $\sigma\in(0,1/2)$ and $\gamma\in(0,1)$. Let $A_1\in\mathscr{A}(\Omega,M,\gamma)$, $A_2\in\mathscr{A}(\Omega,M,0)$ with $A_1=A_2$ on $\partial\Omega$; and $q_1,q_2\in\mathscr{Q}(\Omega,M,\sigma)$. If $U_1,U_2\in H^1(\Omega)$ satisfies $\mathcal{L}_{\widetilde{A}_1,q_1}U_1=0$ and $\mathcal{L}_{\widetilde{A}_2,\overline{q}_2}U_2=0$ and also $U_1|_E=0$, then there exist three positive constants τ_0, C and $\widetilde{\lambda}$ (all depending on n,Ω,M,σ,γ) such that the estimate*

$$
\begin{aligned}
&\left|\left\langle(\widetilde{\Lambda}_1-\widetilde{\Lambda}_2)U_1,U_2\right\rangle_{L^2(\partial\Omega)}\right| \\
&\leqslant C\left(e^{4\tau c}\left\|\Lambda_1^\sharp-\Lambda_2^\sharp\right\|+\tau^{-1/2}\right)\|\overline{g}\|_{H^2(\Omega)} \\
&\qquad + C\tau^{1/2}\left|\log\left|\log\left\|\Lambda_1^\sharp-\Lambda_2^\sharp\right\|\right|\right|^{-\widetilde{\lambda}}\|\overline{g}\|_{H^2(\Omega)}
\end{aligned}
\tag{2.4.14}
$$

holds true for all $\tau\geqslant\tau_0$.

Proof. We start by considering the functions $u_1,u_2\in H^1(\Omega)$, given by theorems 2.2.1 and 2.2.2, respectively; satisfying $\mathcal{L}_{A_1,q_1}u_1=0$ and $\mathcal{L}_{A_2,\overline{q}_2}=0$. Notice also that $u_1|_E=0$. Thus, by identities (2.4.3) and (2.4.4) we have that $U_1=e^{i\omega/2}u_1$ and $U_2=e^{-i\omega/2}u_2$ satisfy

$$
\mathcal{L}_{\widetilde{A}_1,q_1}U_1=0,\quad \mathcal{L}_{\widetilde{A}_2,\overline{q_2}}U_2=0.
$$

Moreover, from (2.4.2) we deduce that $U_1,U_2\in H^1(\Omega)$. Now pick $p>n$ and since $A_1,A_2\in W^{2,\infty}(\Omega)$, we have that $A_1,A_2\in W^{2,p}(\Omega)$. The task now is to compute the norms corresponding to U_1 on the right-hand side of (2.4.6). The estimates for U_2 are similar. From (2.2.13) and (2.4.2), we have

$$
\begin{aligned}
\|U_1\|_{H^1(\Omega)}&=\left\|e^{i\omega/2}u_1\right\|_{H^1(\Omega)}=\left\|e^{i\omega/2}u_1\right\|_{L^2(\Omega)}+\left\|\nabla(e^{i\omega/2}u_1)\right\|_{L^2(\Omega)} \\
&=\left\|e^{i\omega/2}u_1\right\|_{L^2(\Omega)}+\left\|i(\nabla\omega/2)e^{i\omega/2}u_1+e^{i\omega/2}\nabla u_1\right\|_{L^2(\Omega)} \\
&\leqslant C_1\|u_1\|_{H^1(\Omega)}\leqslant C_2\tau e^{\tau c}.
\end{aligned}
\tag{2.4.15}
$$

From (2.2.14) and since $\omega = 0$ on $\partial\Omega$ we obtain

$$\left\| e^{-\tau\xi\cdot x} U_1 \right\|_{L^2(\partial\Omega)} = \left\| e^{-\tau\xi\cdot x} e^{i\omega/2} u_1 \right\|_{L^2(\partial\Omega)} = \left\| e^{-\tau\xi\cdot x} u_1 \right\|_{L^2(\partial\Omega)} \leqslant C_3. \tag{2.4.16}$$

To estimate the term $e^{-\tau\xi\cdot x}(\mathcal{L}_{\widetilde{A}_1,q_1} - \mathcal{L}_{\widetilde{A}_2,\overline{q}_2})U_1$, we first set $V = e^{i\omega/2}(a_1 + r_1 + e^{-\tau(\varphi+i\psi)}e^{\tau l}b)$. Here the fucntions a_1, φ and ψ as in (2.2.12) and r_1, l and b as in Theorem 2.2.1. Thus, from (2.2.15) we have

$$\begin{aligned} &\left\| e^{-\tau\xi\cdot x}(\mathcal{L}_{\widetilde{A}_1,q_1} - \mathcal{L}_{\widetilde{A}_2,\overline{q}_2})U_1 \right\|_{L^2(\Omega)} \\ &= \left\| e^{-\tau(\varphi+i\psi)}(\mathcal{L}_{\widetilde{A}_1,q_1} - \mathcal{L}_{\widetilde{A}_2,\overline{q}_2}) \left[e^{i\omega/2}(e^{\tau(\varphi+i\psi)}(a_1+r_1) - e^{\tau l}b) \right] \right\|_{L^2(\Omega)} \\ &= \left\| e^{-\tau(\varphi+i\psi)}(\mathcal{L}_{\widetilde{A}_1,q_1} - \mathcal{L}_{\widetilde{A}_2,\overline{q}_2}) \left[e^{\tau(\varphi+i\psi)} V \right] \right\|_{L^2(\Omega)} \\ &= \left\| 2\tau(\widetilde{A}_1 - \widetilde{A}_2)\cdot D\rho V + (\mathcal{L}_{\widetilde{A}_1,q_1} - \mathcal{L}_{\widetilde{A}_2,\overline{q}_2})V \right\|_{L^2(\Omega)} \\ &\leqslant C_4 \left(\left\| \widetilde{A}_1 - \widetilde{A}_2 \right\|_{L^\infty(\Omega)} \|V\|_{H^1(\Omega)} + \|V\|_{L^2(\Omega)} \right) \\ &\leqslant C_5 \left(\tau \left\| \widetilde{A}_1 - \widetilde{A}_2 \right\|_{L^\infty(\Omega)} + 1 \right). \end{aligned} \tag{2.4.17}$$

Analogously, from (2.2.17) and (2.2.18) we obtain

$$\|U_2\|_{H^1(\Omega)} \leqslant C_6 \tau e^{\tau c} \|\overline{g}\|_{H^2(\Omega)}, \quad \left\| e^{\tau\xi\cdot x} U_2 \right\|_{L^2(\partial\Omega)} \leqslant C_7 \|\overline{g}\|_{H^2(\Omega)}. \tag{2.4.18}$$

Thus, taking into account that there exists $C_8 > 0$ such that $\tau \leqslant C_8 e^{\tau k}$ for τ large enough and combining the estimates (2.4.15)-(2.4.18) into (2.4.6), we obtain

$$\begin{aligned} &\left| \left\langle (\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2)U_1, U_2 \right\rangle_{L^2(\partial\Omega)} \right| \\ &\leqslant C_9 \left(e^{4\tau c} \left\| \widetilde{\Lambda}_1^\sharp - \widetilde{\Lambda}_2^\sharp \right\| + \tau^{1/2} \left\| \widetilde{A}_1 - \widetilde{A}_2 \right\|_{L^\infty(\Omega)} + \tau^{-1/2} \right) \|\overline{g}\|_{H^2(\Omega)}. \end{aligned} \tag{2.4.19}$$

On the other hand, we fix $q \in \mathbb{R}$ such that $n < p < q$, and consider $t \in (0,1)$ satisfying $1/p = t/2 + (1-t)/q$. Then by elementary interpolation we have

$$\|d(A_1 - A_2)\|_{L^p(\Omega)} \leqslant \|d(A_1 - A_2)\|_{L^2(\Omega)}^t \|d(A_1 - A_2)\|_{L^q(\Omega)}^{1-t}.$$

Hence, Theorem 1.2.4 and (2.4.1), imply that

$$\|A_1 - A_2 - \nabla\omega\|_{C^{0,1-\frac{n}{p}}(\Omega)} \leqslant C_{10} \|dA_1 - dA_2\|_{L^p(\Omega)} \leqslant C_{10} \left| \log \left| \log \left\| \Lambda^\sharp_{A_1,q_1} - \Lambda^\sharp_{A_2,q_2} \right\| \right| \right|^{-t\lambda/2}. \tag{2.4.20}$$

Observe that from (2.4.3) and (2.4.4), we have $\left\| \widetilde{\Lambda}_1^\sharp - \widetilde{\Lambda}_2^\sharp \right\| = \left\| \Lambda_1^\sharp - \Lambda_2^\sharp \right\|$. Moreover, $\widetilde{A}_1 - \widetilde{A}_2 = A_1 - A_2 - \nabla\omega$. So we conclude the proof by combining the above inequality with (2.4.19). □

Corollary 2.4.4 gives us an estimate for the left-hand side of identity (2.4.5). The task now is to isolate $q_1 - q_2$ from its right-hand side. From (2.4.5), we have

$$\begin{aligned}\left|\int_\Omega (q_1 - q_2) U_1 \overline{U}_2\right| &\leqslant \left|\left\langle (\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2) U_1, U_2 \right\rangle_{L^2(\partial\Omega)}\right| \\ &\quad + \left|\int_\Omega (\widetilde{A}_1 - \widetilde{A}_2)\cdot (DU_1 \overline{U}_2 + U_1 \overline{DU}_2)\right| \\ &\quad + \left|\int_\Omega (\widetilde{A}_1 - \widetilde{A}_2)\cdot (\widetilde{A}_1 + \widetilde{A}_2) U_1 \overline{U}_2\right| \\ &\leqslant \left|\left\langle (\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2) U_1, U_2 \right\rangle_{L^2(\partial\Omega)}\right| \\ &\quad + C_1 \left\|\widetilde{A}_1 - \widetilde{A}_2\right\|_{L^\infty} \left(\left\|DU_1 \overline{U}_2 + U_1 \overline{DU}_2\right\|_{L^1(\Omega)} + \left\|U_1 \overline{U}_2\right\|_{L^1(\Omega)}\right).\end{aligned} \tag{2.4.21}$$

Recall that $U_1 = e^{i\omega/2} u_1$ and $U_2 = e^{-i\omega/2} u_2$, where $u_1, u_2 \in H^1(\Omega)$ satisfy $\mathcal{L}_{A_1,q_1} u_1 = 0$ and $\mathcal{L}_{A_2,\overline{q}_2} u_2 = 0$, respectively. Hence, from (2.4.2), (2.3.2)-(2.3.5) and an easy computation we have

$$\begin{aligned}&\left\|DU_1 \overline{U}_2 + U_1 \overline{DU}_2\right\|_{L^1(\Omega)} + \left\|U_1 \overline{U}_2\right\|_{L^1(\Omega)} \\ &\quad \leqslant C_2 \left(\left\|Du_1 \overline{u}_2 + u_1 \overline{Du}_2\right\|_{L^1(\Omega)} + \left\|u_1 \overline{u}_2\right\|_{L^1(\Omega)}\right) \leqslant C_3 \tau.\end{aligned} \tag{2.4.22}$$

We consider now the functions $u_1, u_2 \in H^1(\Omega)$ given by theorem C.1.1 and 2.2.2, respectively. As in (2.2.12) and (2.3.1), we denote $a_1 = e^{\Phi_1}$ and $a_2 = e^{\Phi_2} g$, where g is any smooth function satisfying (2.2.9).Thus, we have the identity

$$\begin{aligned}\int_\Omega e^{i\omega}(q_1 - q_2) a_1 \overline{a}_2 &= \int_\Omega (q_1 - q_2) U_1 \overline{U}_2 \\ &\quad - \int_\Omega e^{i\omega}(q_1 - q_2)\left[a_1 \overline{r}_2 + r_1 \overline{a}_2 + r_1 \overline{r}_2 + e^{-\tau(\varphi + i\psi)} e^{\tau l} b(\overline{a}_2 + \overline{r}_2)\right],\end{aligned}$$

and combining (2.4.21)-(2.4.22) with (2.2.11), (2.2.16) and (2.3.4); we obtain

$$\begin{aligned}\left|\int_\Omega e^{i\omega}(q_1 - q_2) a_1 \overline{a}_2\right| &\leqslant \left|\left\langle (\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2) U_1, U_2 \right\rangle_{L^2(\partial\Omega)}\right| \\ &\quad + C_4 \tau \left\|\widetilde{A}_1 - \widetilde{A}_2\right\|_{L^\infty} + C_5 \tau^{-1}.\end{aligned}$$

This inequality, (2.4.20) and Corollary 2.4.4, imply that there exist two positive constants τ_0 and C_6 such that

$$\begin{aligned}&\left|\int_\Omega e^{i\omega}(q_1 - q_2) a_1 \overline{a}_2\right| \leqslant C_6 \left\|\overline{g}\right\|_{H^2(\Omega)} \\ &\quad \times \left(e^{4\tau c} \left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\| + \tau \left|\log\left|\log\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\|\right|\right|^{-\widetilde{\lambda}} + \tau^{-1/2}\right),\end{aligned} \tag{2.4.23}$$

for all $\tau \geqslant \tau_0$.

Remark 2.4.5. *The following result was proved in [44] (see Lemma 4.6 in [44] and also Lema 2.1 in [46]). Let $\xi_0 \in \mathbb{C}^n$ such that $\Re\xi_0 \cdot \Im\xi_0 = 0$ and $|\Re\xi_0| = |\Im\xi_0| = 1$. If $W \in L^\infty(\mathbb{R}^n)$ then there exists a solution $\Phi \in L^\infty(\mathbb{R}^n)$ of the equation*

$$\xi_0 \cdot \nabla\Phi + i\xi_0 \cdot W = 0.$$

Moreover, there exists $C > 0$ such that

$$\|\Phi\|_{L^\infty(\mathbb{R}^n)} \leqslant C\,\|W\|_{L^\infty(\mathbb{R}^n)}\,. \tag{2.4.24}$$

Proposition 2.4.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$. Consider three positive constants M, $\sigma \in (0,1/2)$ and $\gamma \in (0,1)$. Let $A_1 \in \mathscr{A}(\Omega, M, \gamma)$, $A_2 \in \mathscr{A}(\Omega, M, 0)$ with $A_1 = A_2$ on $\partial\Omega$; and $q_1, q_2 \in \mathscr{Q}(\Omega, M, \sigma)$. Consider any smooth function g satisfying $(\xi + i\zeta)\cdot\nabla g = 0$ (see (2.2.9)). Then there exist two positive constants τ_0 and C (both depending on $n, \Omega, M.\sigma, \gamma$) such that*

$$\left|\int_\Omega (q_1 - q_2)\overline{g}\right| \leqslant C\left|\log\left|\log\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\|\right|\right|^{-\frac{\widetilde{\lambda}}{3}}\|\overline{g}\|_{H^2(\Omega)}\,, \tag{2.4.25}$$

provided that $\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\| \leqslant e^{-e^{(8c\tau_0)^{\frac{3}{2}\widetilde{\lambda}^{-1}}}}$.

Proof. We start with the following identity

$$\int_\Omega (q_1 - q_2)\overline{g} = \int_\Omega (1 - e^{\Phi_1 + \overline{\Phi}_2 + i\omega})(q_1 - q_2)\overline{g} + \int_\Omega e^{i\omega}(q_1 - q_2)a_1\overline{a}_2, \tag{2.4.26}$$

From (2.2.2) and (2.2.7), we have

$$(\xi + i\zeta)\cdot\nabla(\Phi_1 + \overline{\Phi}_2) + i(\xi + i\zeta)\cdot(A_1 - A_2) = 0,$$

which imply that

$$(\xi + i\zeta)\cdot\nabla(\Phi_1 + \overline{\Phi}_2 + i\omega) + i(\xi + i\zeta)\cdot(A_1 - A_2 - \nabla\omega) = 0.$$

Thus, by estimate (2.4.24) from Remark 2.4.5, we get

$$\left\|\Phi_1 + \overline{\Phi}_2 + i\omega\right\|_{L^\infty(\Omega)} \leqslant C_1\,\|A_1 - A_2 - \nabla\omega\|_{L^\infty(\Omega)}\,.$$

We can now estimate the first term of the right-hand side of (2.4.26). Using the inequality

$$\left|e^a - e^b\right| \leqslant |a - b|\,e^{\max\{\Re a, \Re b\}}\ , \quad a, b \in \mathbb{C},$$

we obtain

$$\begin{aligned}\left|\int_\Omega (1 - e^{\Phi_1 + \overline{\Phi}_2 + i\omega})(q_1 - q_2)\overline{g}\right| &= \left|\int_\Omega (e^0 - e^{\Phi_1 + \overline{\Phi}_2 + i\omega})(q_1 - q_2)\overline{g}\right| \\ &\leqslant \left\|(\Phi_1 + \overline{\Phi}_2 + i\omega)e^{\max\left\{0, \Re(\Phi_1 + \overline{\Phi}_2 + i\omega)\right\}}\right\|_{L^\infty(\Omega)}\int_\Omega |(q_1 - q_2)\overline{g}| \\ &\leqslant C_2\,\|A_1 - A_2 - \nabla\omega\|_{L^\infty(\Omega)}\,\|\overline{g}\|_{L^2(\Omega)}\,.\end{aligned}$$

Taking into account (2.4.26), (2.4.20) and (2.4.23), we have that there exist $\tau_0 > 0$ and $C_3 > 0$ such that

$$\begin{aligned}\left|\int_\Omega (q_1 - q_2)\overline{g}\right| &\leqslant C_3 \left\|\overline{g}\right\|_{H^2(\Omega)} \\ &\times \left(e^{4\tau c}\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\| + \tau\left|\log\left|\log\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\|\right|\right|^{-\widetilde{\lambda}} + \tau^{-1/2}\right).\end{aligned} \tag{2.4.27}$$

We conclude the proof by taking

$$\tau = \frac{1}{8c}\left|\log\left|\log\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\|\right|\right|^{\frac{2}{3}\widetilde{\lambda}} \geqslant \tau_0,$$

which is true whenever

$$\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\| \leqslant e^{-e^{(8c\tau_0)^{\frac{3}{2}\widetilde{\lambda}^{-1}}}}.$$

□

2.4.1 Proof of Theorem 1.1.5

We begin by considering the notation in Radon transform introduced in Theorem 1.2.4 and proceed analogously as in its proof. The estimate (2.4.25) from Proposition 2.4.6, imply that

$$\left|\int_{\mathbb{R}} \widetilde{g}(s)(\boldsymbol{R}\left[\chi_\Omega(q_1 - q_2)\right])(s,\theta)ds\right| \leqslant C_1\left|\log\left|\log\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\|\right|\right|^{-\frac{\widetilde{\lambda}}{3}} \left\|\widetilde{g}\right\|_{H^2(\mathbb{R})},$$

for all $\theta \in M$. The set M is defined by (2.3.18). From this inequality, we deduce that

$$\left\|\boldsymbol{R}\left(\chi_\Omega(q_1 - q_2)\right)\right\|_{H^{-2}(\mathbb{R};L^\infty(M))} \leqslant C_2\left|\log\left|\log\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\|\right|\right|^{-\frac{\widetilde{\lambda}}{3}}.$$

On the other hand, from (2.3.15), we get

$$\left\|\boldsymbol{R}\left(\chi_\Omega(q_1 - q_2)\right)\right\|_{H^{\frac{n-1}{2}}(\mathbb{R};L^2(M))} \leqslant C_3\left\|\chi_\Omega(q_1 - q_2)\right\|_{L^2(\mathbb{R}^n)} \leqslant C_4.$$

Thus, by a standard interpolation between the spaces $H^{-2}(\mathbb{R};L^\infty(M))$ and $H^{\frac{n-1}{2}}(\mathbb{R};L^2(M))$, we obtain

$$\begin{aligned}&\left\|\boldsymbol{R}\left(\chi_\Omega(q_1 - q_2)\right)\right\|_{L^2(\mathbb{R};L^{(n+3)/2}(M))} \\ &\qquad\leqslant C_3\left|\log\left|\log\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\|\right|\right|^{-\frac{\widetilde{\lambda}}{3}(n-1)/(n+3)}.\end{aligned} \tag{2.4.28}$$

We are now in the position to apply Theorem 2.3.4 to the function $\chi_\Omega(q_1 - q_2)$. Let us verify its three conditions. Since Ω is bounded, the supporting condition (b) is satisfied for some $y_0 \in \mathbb{R}^n$. From the above estimate, there exists $\beta \in (0,1)$ such that the condition (a) is satisfied for any $\alpha > 0$. Thus, by taking $\alpha > 0$ large enough it follows that

$\operatorname{supp}(\chi_\Omega(q_1 - q_2)) \subset G$. Since $q_1, q_2 \in H^\sigma(\mathbb{R}^n)$ and $\chi_\Omega \in H^{1/2-\sigma}(\mathbb{R}^n)$ (for this last fact see [19]), the condition (c) is satisfied for $p = 2$ and $0 < \lambda < 1/2$. For convenience, we set $q = \chi_\Omega(q_1 - q_2)$. Then Theorem 2.3.4 ensures that there exists $C_4 > 0$ such that

$$\|q\|_{L^2(\mathbb{R}^n)} \leqslant C_4 \left|\log \int_{-\alpha}^{\alpha} (1+|s|)^n \left\|\boldsymbol{R}_{y_0} q(s,\cdot)\right\|_{L^1(\Gamma)} ds\right|^{-\lambda/2}. \tag{2.4.29}$$

Analogously to the proof of stability estimate for the magnetic potentials, here the set Γ is where we have the control of the Radon transform on the θ-variable. In our case (see the estimate (2.4.28)) we have the control on M. Now we set

$$L = \sup_{\theta \in M} \|(1 + |\cdot - \langle\theta, y_0\rangle|)^n\|_{L^2(|s| \leqslant \alpha + |y_0|)}$$

and denote by $|M|$ the measure of M. Then the inequality (2.4.28), Fubini's theorem and Hölder's inequality applied twice, and a repetition of the arguments at the end of the proof of Theorem 1.2.4 give us

$$\begin{aligned} &\int_{-\alpha}^{\alpha} (1+|s|)^n \left\|\boldsymbol{R}_{y_0} q(s,\cdot)\right\|_{L^1(M)} ds \\ &\leqslant C_5 \left|\log\left|\log\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\|\right|\right|^{-\frac{\tilde\lambda}{3}(n-1)/(n+3)}. \end{aligned}$$

We conclude the proof by taking logarithms on both sides of the above inequality and taking into account the estimate (2.4.29).

2.5 Identifiability for the magnetic field and the electric potential

2.5.1 Proof of Theorem 1.1.1

The proof is an immediate consequence of the previous stability estimates for the magnetic fields and the electric potentials because it is just the qualitative version of what we have proved in the previous sections. By hypothesis $\Lambda_1^\sharp = \Lambda_2^\sharp$, then $\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\| = 0$. Hence, we can follow all previous computations with $\left\|\Lambda_1^\sharp - \Lambda_2^\sharp\right\|$ replaced by 0. Thus, for the magnetic fields the estimates (2.3.21)-(2.3.22) imply that

$$\partial_{x_i}\widetilde{A}_j - \partial_{x_j}\widetilde{A}_i = 0, \quad i, j = 1, \dots, n,$$

where $\widetilde{A} = \chi_\Omega(A_1 - A_2)$. Since $A_1 = A_2$ on $\partial\Omega$ and from the above identity, it follows that $dA_1 = dA_2$ in Ω. Analogously for the electric potentials, from (2.4.28)-(2.4.29), it follows immediately that $q_1 = q_2$ in Ω.

Chapter 3

An IBVP for a magnetic Schrödinger operator with local data

In this chapter we prove the identifiability results stated in Theorem 1.2.1 and also the corresponding stability estimates for the magnetic fields and the electric potentials stated in the theorems 1.2.4 and 1.2.5. We start by proving an integral estimate relating the local Cauchy data sets with the magnetic and electric potentials in Ω. Subsequently, in order to subtract the information about $dA_1 - dA_2$ and $q_1 - q_2$ coded in the integral estimate we construct special solutions $u \in H^1(\Omega)$ of the magnetic Schrödinger operator $\mathcal{L}_{A,q}u = 0$ wit the vanishing condition on inaccessible part of the boundary Γ_0. This step leads us to obtain Fourier transforms, one for the difference of the magnetic fields $dA_1 - dA_2$ and another for the difference of the electric potentials $q_1 - q_2$, plus some error terms. At this point, to end up the proof of Theorem 1.2.1, we use the invertibility of the Fourier transform plus the Riemann-Lebesgue Lemma. To end up the proof of theorems 1.2.4 and 1.2.5, we use the Fourier transform and a quantitative estimate for the Riemann-Lebesgue Lemma derived in [24]. Additionally, in the latter theorem, we have to use an extra argument: a Hodge decomposition derived by Caro and Pohjola [8].

3.1 Preliminaries - Definitions and notations

Here we introduce the definitions of the local boundary map T_r^Γ, the local linear map $N_{A,q}^\Gamma$, the space $H^1(\Omega,\Gamma)$, the admissible class of the magnetic potentials $\mathcal{A}(\Omega, M, s)$, the admissible class of the electric potentials $\mathcal{Q}(\Omega, M, s)$ and the pseudo-distance between local-data Cauchy sets $\mathrm{dist}(\,\cdot\,,\,\cdot\,)$. The space $H^1(\Omega,\Gamma)$ is defined by density as

$$H^1(\Omega,\Gamma) := \overline{\left\{u \in C^\infty(\overline{\Omega}) \,:\, u|_{\Gamma_0} = 0\right\}}^{H^1(\Omega)}, \tag{3.1.1}$$

where $\overline{E}^{H^1(\Omega)}$ denotes the closure of the set E in the topology of $H^1(\Omega)$, with the norm $H^1(\Omega)$ restricted to E. Thus, we define the local boundary map as

$$T_r^\Gamma : H^1(\Omega,\Gamma) \to H^1(\Omega,\Gamma)/H_0^1(\Omega), \quad T_r^\Gamma u = [u]\,, \tag{3.1.2}$$

where $[u]$ denotes the equivalence class of $u \in H^1(\Omega,\Gamma)$ in the quotient space $H^1(\Omega,\Gamma)/H_0^1(\Omega)$. The local linear map $N_{A,q}^\Gamma : H^1(\Omega,\Gamma)/H_0^1(\Omega) \to \left(H^1(\Omega,\Gamma)/H_0^1(\Omega)\right)^*$ is defined by

$$\left\langle N_{A,q}^\Gamma\,[u]\,,[g]\right\rangle = \int_\Omega Du\cdot\overline{Dv} + A\cdot(Du\overline{v} + u\overline{Dv}) + (A^2+q)u\overline{v}, \tag{3.1.3}$$

for all $u \in H^1(\Omega,\Gamma)$ satisfying $\mathcal{L}_{A,q}u = 0$ (in Ω) and for any $v \in [g]$ with $g \in H^1(\Omega,\Gamma)$. Observe that $N_{A,q}^\Gamma$ makes sense for equivalence class of functions $u \in H^1(\Omega)$ satisfying $\mathcal{L}_{A,q}u = 0$ in Ω. With these definitions and notations at hand, the local Cauchy data set can also be written as

$$C_{A,q}^\Gamma = \left\{([u]\,, N_{A,q}^\Gamma\,[u]) : u \in H^1(\Omega,\Gamma),\, \mathcal{L}_{A,q}u = 0 \text{ in } \Omega\right\}, \tag{3.1.4}$$

see (1.2.5). Following [8], we introduce the pseudo-distance $\operatorname{dist}(\,\cdot\,,\,\cdot\,)$, inspired in the Hausdorff distance. Let $A_1, A_2 \in L^\infty(\Omega;\mathbb{C}^n)$ be two magnetic potentials and let $q_1, q_2 \in L^\infty(\Omega)$ be two electric potentials. Given $(f,g) \in C_{A_j,q_j}^\Gamma$ with $j = 1,2$, we set

$$\begin{aligned} &I((f,g);C_{A_k,q_k}^\Gamma) \\ &\quad = \inf_{(f_k,g_k)\in C_{A_k,q_k}^\Gamma} \left[\|f-f_k\|_{H^1(\Omega,\Gamma)/H_0^1(\Omega)} + \|g-g_k\|_{(H^1(\Omega,\Gamma)/H_0^1(\Omega))^*}\right], \end{aligned}$$

with $k = 1,2$. Then, the pseudo-distance between C_{A_1,q_1}^Γ and C_{A_2,q_2}^Γ is defined by

$$\operatorname{dist}(C_{A_1,q_1}^\Gamma, C_{A_2,q_2}^\Gamma) = \max_{\substack{j,k=1,2\\ j\neq k}} \sup_{\substack{(f_j,g_j)\in C_{A_j,q_j}^\Gamma\\ \|f_j\|_{H^1(\Omega,\Gamma)/H_0^1(\Omega)}=1}} I((f_j,g_j);C_{A_k,q_k}^\Gamma). \tag{3.1.5}$$

Throughout this chapter we denote by $C_i^\Gamma = C_{A_i,q_i}^\Gamma$ the local Cauchy data sets with respect to the magnetic potential A_i and the electric potential q_i with $i = 1,2$.

3.2 Relating the local Cauchy data sets with the magnetic and electric potentials in Ω

In this section we state an integral estimate which involves a relation between the magnetic and electric potentials and the distance between their corresponding Cauchy data sets. This integral identity was implicitly proved in [28], see Proposition 3.2 therein.

Lemma 3.2.1. *Let Ω be an open bounded set. Let $A_1, A_2 \in L^\infty(\Omega;\mathbb{C}^n)$ and $q_1, q_2 \in L^\infty(\Omega;\mathbb{C})$. Let $U_1, U_2 \in H^1(\Omega)$ be functions satisfying in Ω: $\mathcal{L}_{A_1,q_1}U_1 = 0$ and $\mathcal{L}_{\overline{A_2},\overline{q_2}}U_2 = 0$. Then the following identity holds true:*

$$\begin{aligned}&\left\langle N_{A_1,q_1}[U_1], \overline{T_r U_2}\right\rangle - \overline{\left\langle N_{\overline{A_2},\overline{q_2}}[U_2], \overline{T_r U_1}\right\rangle}\\ &= \int_\Omega \left[(A_1 - A_2)\cdot\left(DU_1\overline{U_2} + U_1\overline{DU_2}\right) + \left(A_1^2 - A_2^2 + q_1 - q_2\right)U_1\overline{U_2}\right] dx.\end{aligned}$$

Corollary 3.2.2. *Consider all conditions from Lemma 3.2.1. Consider also $s \in (0, 1/2)$. Let $U_1, U_2 \in H^1(\Omega)$ be functions satisfying in Ω: $\mathcal{L}_{A_1,q_1}U_1 = 0$ with $U_1|_{\Gamma_0} = 0$ and $\mathcal{L}_{\overline{A_2},\overline{q_2}}U_2 = 0$ with $U_2|_{\Gamma_0} = 0$. Then there exists a positive constant C (depending on $n, \Omega, \|A_j\|_{L^\infty \cap B_s^{2,\infty}}, \|q_j\|_{L^\infty}; j = 1,2$) such that*

$$\begin{aligned}&\left|\int_\Omega (A_1 - A_2)\cdot\left(DU_1\overline{U_2} + U_1\overline{DU_2}\right) + \left(A_1^2 - A_2^2 + q_1 - q_2\right)U_1\overline{U_2}\right|\\ &\qquad \leqslant C dist\left(C_1^\Gamma, C_2^\Gamma\right)\|U_1\|_{H^1(\Omega)}\|U_2\|_{H^1(\Omega)},\end{aligned} \tag{3.2.1}$$

where C_j^Γ denotes the local Cauchy data set C_{A_j,q_j}^Γ, $j = 1, 2$.

Remark 3.2.3. *Corollary 3.2.2 was proved for full data case in Proposition 2.1 in [8]. The proof for local data follows easily by taking into account that $U_1|_{\Gamma_0} = 0$, $U_2|_{\Gamma_0} = 0$ and the definition of $dist\left(C_1^\Gamma, C_2^\Gamma\right)$ given by (3.1.5).*

3.3 Construction of special solutions - CGO solutions

In order to exploit the information encoded in the integral inequality (3.2.1), we will construct special solutions $u \in H^1(\Omega)$ of the magnetic Schrödinger operator $\mathcal{L}_{A,q}u = 0$ with the desired vanishing condition $u|_{\Gamma_0} = 0$. As a first approach to do that, we state a known result about the existence of solutions on a bounded set but do not require the vanishing condition. The main result in this section is Theorem 3.3.1.

Theorem 3.3.1. *Let $V \subset \mathbb{R}^n$ be a bounded open set. Consider $s \in (0, 1/2)$. Let $A \in L^\infty \cap B_s^{2,\infty}(\mathbb{R}^n;\mathbb{C}^n)$ and $q \in L^\infty(\mathbb{R}^n;\mathbb{C})$ such that $\operatorname{supp} A \subset \overline{V}$ and $\operatorname{supp} q \subset \overline{V}$. Consider $\rho \in \mathbb{C}^n$ such that $\rho\cdot\rho = 0$ and $\rho = \rho_0 + \rho_\tau$ with ρ_0 being independent of some large parameter $\tau > 0$, $|\Re\rho_0| = |\Im\rho_0| = 1$ and $\rho_\tau = \mathcal{O}(\tau^{-1})$ as $\tau \mapsto \infty$. Then there exist two positive constants C and τ_0 (both depending on $n, V, s, \|A\|_{L^\infty\cap B_s^{2,\infty}}, \|q\|_{L^\infty}$) ; and a solution $u \in H^1(V)$ to the equation $\mathcal{L}_{A,q}u = 0$ in V of the form*

$$u(x,\rho;\tau) = e^{\tau\rho\cdot x}\left(e^{\Phi^\sharp(x,\rho_0;\tau)} + r(x,\rho;\tau)\right)$$

with the following properties:

(i) The function $\Phi^\sharp(\cdot,\rho_0;\tau)\in C^\infty(\mathbb{R}^n)$ *and satisfies for all* $\alpha\in\mathbb{N}^n$

$$\left\|\partial^\alpha\Phi^\sharp(\cdot,\rho_0;\tau)\right\|_{L^\infty(\mathbb{R}^n)}\leqslant C\tau^{|\alpha|/(s+2)}\left\|A\right\|_{L^\infty(\mathbb{R}^n)},\quad \tau\geqslant\tau_0. \tag{3.3.1}$$

(ii) The function $r(\cdot,\rho_0;\tau)\in H^1(V)$ *and satisfies*

$$\left\|\partial^\alpha r(\cdot,\rho_0;\tau)\right\|_{L^2(V)}\leqslant C\,\tau^{|\alpha|-s/(s+2)},\quad |\alpha|\leqslant 1. \tag{3.3.2}$$

(iii) If we define by $\kappa:=\sup_{x\in\overline{V}}|x|$ *then* u *satisfies*

$$\left\|u\right\|_{H^1(V)}\leqslant C\,e^{\tau\kappa|\rho|}. \tag{3.3.3}$$

Moreover, if we denote by $\Phi(\cdot;\rho_0)=(\rho_0\cdot\nabla)^{-1}(-i\rho_0\cdot A)\in L^\infty(\mathbb{R}^n)$, *the function satisfying the following equation in* $\mathbb{R}^n$

$$\rho_0\cdot\nabla\Phi+i\rho_0\cdot A=0 \tag{3.3.4}$$

then

$$\left\|\Phi(\cdot;\rho_0)\right\|_{L^\infty(\mathbb{R}^n)}\leqslant C\left\|A\right\|_{L^\infty(\mathbb{R}^n)}. \tag{3.3.5}$$

Finally, for every $\chi\in C_0^\infty(\mathbb{R}^n)$ *we have*

$$\left\|\chi(\Phi^\sharp(\cdot,\rho_0;\tau)-\Phi(\cdot;\rho_0))\right\|_{L^2(\mathbb{R}^n)}\leqslant C\tau^{-s/(s+2)}\left\|A\right\|_{L^\infty(\mathbb{R}^n)}, \tag{3.3.6}$$

where the constant C *also depends on* χ.

Remark 3.3.2. *This theorem is a summary of two known results. On one hand, the existence of* $u\in H^1(V)$ *satisfying* $\mathcal{L}_{A,q}u=0$ *in* V *(with* $A\in L^\infty$ *and* $q\in L^\infty$*) was proved by Krupchyk and Uhlmann in [28]. On the other hand, when* $A\in L^\infty\cap B_s^{2,\infty}$ *and* $q\in L^\infty$, *the corresponding estimates for* $\Phi^\sharp(\cdot,\rho_0;\tau),\Phi(\cdot;\rho_0)$ *and* $r(\cdot,\rho_0;\tau)$ *have been taken from Proposition* 2.6 *in [28] and the section* 3 *in [8]. For these reasons we only give the main ideas of the proof with the repetition of the relevant material from [8] and [28], thus making our exposition self-contained.*

Proof. In [28] it it was established that for all $A\in L^\infty(V)$ and for all $q\in L^\infty(V)$ there exists a function $u\in H^1(V)$ satisfying $\mathcal{L}_{A,q}u=0$ in V of the form

$$u(x)=e^{\tau\rho\cdot x}(a+r), \tag{3.3.7}$$

where $\rho\in\mathbb{C}^n$ with $\rho\cdot\rho=0$, τ is a large positive parameter, a is a smooth function solving a transport equation, see (3.3.10), and r satisfies a remainder equation, see (3.3.11). The construction involves basically two arguments. The first argument concerns a mollification procedure for the magnetic potential A. More precisely: consider $\varphi\in C_0^\infty(\mathbb{R}^n)$ such that $0\leqslant\varphi\leqslant 1$ and $\operatorname{supp}\varphi\subset\overline{B_1(0)}$, where $\overline{B_1(0)}$ denotes the closure of the ball in $\mathbb{R}^n$ of radius 1 centered at the origin. For each $\delta>0$ we define $\varphi_\delta(x)=\delta^{-n}\varphi(x/\delta)$ and we set

$A^\sharp_\delta = A * \varphi_\delta$ which belongs to $C_0^\infty(\mathbb{R}^n;\mathbb{C}^n)$. Then, there exists a positive constant $C_1 > 0$ (depending on V and n) such that

$$\left\| A - A^\sharp_\delta \right\|_{L^2(\mathbb{R}^n)} \leqslant C_1\, \delta^s \left\| A \right\|_{B_s^{2,\infty}(\mathbb{R}^n)} \tag{3.3.8}$$

and for each $\alpha \in \mathbb{N}^n$ we have

$$\left\| \partial^\alpha A^\sharp_\delta \right\|_{L^\infty(\mathbb{R}^n)} \leqslant C_2\, \delta^{-|\alpha|} \left\| A \right\|_{L^\infty(\mathbb{R}^n)}. \tag{3.3.9}$$

See Section 3 in [8] for more details. The second argument involves the use of a Carleman estimate for the Laplacian derived by Salo and Tzou [43]. This Carleman estimate is between $H^1(V)$ and its dual space $H^{-1}(V)$. An easy computation show us that a function $u \in H^1(V)$ of the form (3.3.7) is a solution of the equation $\mathcal{L}_{A,q}u = 0$ if the following identity

$$\begin{aligned} 0 = \tau^{-2}\mathcal{L}_{A,q}a - \tau^{-1}&\left(2i\rho_\tau\cdot Da + 2i\rho_0\cdot(A - A^\sharp_\delta)a + 2i\rho_\tau\cdot Aa\right) \\ &- \tau^{-1}\left(2i\rho_0\cdot Da + 2i\rho_0\cdot A^\sharp_\delta a\right) + e^{-\tau\rho\cdot x}\tau^{-2}\mathcal{L}_{A,q}(e^{\tau\rho\cdot x}r), \end{aligned}$$

holds true in $H^{-1}(V)$. This allow us to consider the function a being a solution in $\mathbb{R}^n$ of the equation

$$\rho_0\cdot\nabla a + i\rho_0\cdot A^\sharp_\delta a = 0 \tag{3.3.10}$$

and the function r satisfying the following equation in $H^{-1}(V)$

$$\begin{aligned} &e^{-\tau\rho\cdot x}\tau^{-2}\mathcal{L}_{A,q}(e^{\tau\rho\cdot x}r) \\ &\quad = -\tau^{-2}\mathcal{L}_{A,q}a + 2i\tau^{-1}\left(\rho_1\cdot Da + \rho_0\cdot(A - A^\sharp_\delta)a + \rho_1\cdot Aa\right). \end{aligned} \tag{3.3.11}$$

The equation (3.3.10) and (3.3.11) are called transport and remainder equations, respectively. The transport equation (3.3.10) can be solved as follows. If we make the ansatz $a = e^{\Phi^\sharp}$ then $\Phi^\sharp$ satisfies the equation in $\mathbb{R}^n$

$$\rho_0\cdot\nabla\Phi^\sharp + i\rho_0\cdot A^\sharp_\delta = 0. \tag{3.3.12}$$

This equation is easy to solve because the condition $\rho_0\cdot\rho_0 = 0$ imply that $\Re\rho_0\cdot\Im\rho_0 = 0$ and $|\Re\rho_0| = |\Im\rho_0|$ and then the operator $\rho_0\cdot\nabla$ becomes a $\partial_{\overline{z}}$ operator, where for each $x\in\mathbb{R}^n$ we have considered the complex variable $z(x) = \Re\rho_0\cdot x + i\Im\rho_0\cdot x$. Thus, the function $\Phi^\sharp = (\rho_0\cdot\nabla)^{-1}(-i\rho_0\cdot A^\sharp_\delta)$ belongs to $C^\infty(\mathbb{R}^n)$ and satisfies (3.3.12). Moreover, from (3.3.9), we have that for all $\alpha\in\mathbb{N}^n$ there exists a constant $C_3 > 0$ such that

$$\left\| \partial^\alpha \Phi^\sharp \right\|_{L^\infty(\mathbb{R}^n)} \leqslant C_3 \left\| \partial^\alpha A^\sharp_\delta \right\|_{L^\infty(\mathbb{R}^n)} \leqslant C_3\, \delta^{-|\alpha|} \left\| A \right\|_{L^\infty(\mathbb{R}^n)}. \tag{3.3.13}$$

For more details about the solvability of (3.3.12), see for example Lema 4.6 in [44]. For similar reasons, the function $\Phi(\cdot;\rho_0) = (\rho_0\cdot\nabla)^{-1}(-i\rho_0\cdot A) \in L^\infty(\mathbb{R}^n)$ solves the following equation in $\mathbb{R}^n$

$$\rho_0\cdot\nabla\Phi + i\rho_0\cdot A = 0.$$

and we have the following estimate

$$\|\Phi(\cdot;\rho_0)\|_{L^\infty(\mathbb{R}^n)} \leqslant C_4 \|A\|_{L^\infty(\mathbb{R}^n)},$$

where the constant $C_4 > 0$ only depend on V and n. Also from (3.3.8), for every $\chi \in C_0^\infty(\mathbb{R}^n)$ there exist a constant $C_5 > 0$ (depending on Ω, n and χ) such that

$$\left\|\chi(\Phi^\sharp(\cdot,\rho_0) - \Phi(\cdot;\rho_0))\right\|_{L^2(\mathbb{R}^n)} \leqslant C_5\, \delta^s \|A\|_{B_s^{2,\infty}(\mathbb{R}^n)},$$

see the section 3 in [8] for more details. Now we explain the solvability of the remainder equation (3.3.11). We start by setting

$$w = -\tau^{-2}\mathcal{L}_{A,q}a + 2i\tau^{-1}\left(\rho_\tau \cdot Da + \rho_0 \cdot (A - A_\delta^\sharp)a + \rho_\tau \cdot Aa\right).$$

Then by Proposition 2.3 in [28], there exists $r \in H^1(V)$ a solution of (3.3.11) and two positive constants C_6 and τ_0 such that

$$\|r\|_{H^1_{scl}(V)} \leqslant C_6\, \tau \|w\|_{H^{-1}_{scl}(V)}, \tag{3.3.14}$$

for all $\tau \geqslant \tau_0$. Here the semi-classical norms are defined by

$$\|r\|^2_{H^1_{scl}(V)} = \|r\|^2_{L^2(V)} + \left\|\tau^{-1}\nabla r\right\|^2_{L^2(V)},$$

$$\|w\|_{H^{-1}_{scl}(V)} = \sup_{0\neq\phi\in C_0^\infty(V)} \frac{\langle w, \phi\rangle_{L^2(V)}}{\|w\|_{H^1_{scl}(V)}}.$$

If we define $\kappa := \sup_{x\in\overline{V}} |x|$ then from (3.3.13) and by taking $\delta = \tau^{-1/(s+2)}$ into (3.3.9), we get

$$\begin{aligned}\|w\|_{H^{-1}_{scl}(V)} &\leqslant C_7\, e^{\kappa\|A\|_{L^\infty}} \tau^{-(2s+2)/(s+2)} \\ &\quad \times \left(1 + \|A\|_{L^\infty} + \|A\|^2_{L^\infty} + \|q\|_{L^\infty} + \|A\|_{B_s^{2,\infty}}\right).\end{aligned}$$

Combining the above inequality with (3.3.14), we obtain

$$\begin{aligned}\|r\|_{H^1_{scl}(V)} &\leqslant C_8\, e^{\kappa\|A\|_{L^\infty}} \tau^{-s/(s+2)} \\ &\quad \times \left(1 + \|A\|_{L^\infty} + \|A\|^2_{L^\infty} + \|q\|_{L^\infty} + \|A\|_{B_s^{2,\infty}}\right).\end{aligned} \tag{3.3.15}$$

By similar computations, we obtain

$$\begin{aligned}\|u\|_{H^1(V)} &\leqslant C_9\, e^{\tau\kappa|\rho|} e^{C\|A\|_{L^\infty(V)}} \\ &\quad \times \left(1 + \|A\|_{L^\infty} + \|A\|^2_{L^\infty} + \|q\|_{L^\infty} + \|A\|_{B_s^{2,\infty}}\right).\end{aligned} \tag{3.3.16}$$

This complete the main ideas of the proof. $\square$

Remark 3.3.3 (Estimates for the identifiability result). *For our identifiability result stated in Theorem 1.2.1, we are only assuming that $A \in L^\infty(\Omega; \mathbb{C}^n)$. In this case, we can obtain similar estimates (3.3.1)-(3.3.6). These estimates are enough to prove Theorem 1.2.1 (see Proposition* 2.6 *in [28]) and can be stated as follows. There exist two positive constants C and τ_0 such that: the function $\Phi^\sharp(\cdot, \rho_0; \tau) \in C^\infty(\mathbb{R}^n)$ and satisfies for all $\alpha \in \mathbb{N}^n$ and for all $\lambda \in (0, 1/2)$*

$$\left\| \partial^\alpha \Phi^\sharp(\cdot, \rho_0; \tau) \right\|_{L^\infty(\mathbb{R}^n)} \leqslant C\tau^{\lambda|\alpha|}, \quad \tau \geqslant \tau_0. \tag{3.3.17}$$

The function $r(\cdot, \rho_0; \tau) \in H^1(V)$ and satisfies for all $|\alpha| \leqslant 1$

$$\|\partial^\alpha r(\cdot, \rho_0; \tau)\|_{L^2(V)} \leqslant C\,\tau^{|\alpha|}, \quad \tau \geqslant \tau_0. \tag{3.3.18}$$

The estimate (3.3.5) is the same. Finally, for every $\chi \in C_0^\infty(\mathbb{R}^n)$ we have

$$\lim_{\tau \to \infty} \left\| \chi(\Phi^\sharp(\cdot, \rho_0; \tau) - \Phi(\cdot; \rho_0)) \right\|_{L^2(\mathbb{R}^n)} = 0. \tag{3.3.19}$$

where the constant C also depends on χ. We will not use these estimates until Section 3.8 to prove Theorem 1.2.1.

3.4 Construction of special solutions vanishing on the inaccessible part of the boundary Γ_0

In this section, we will use Theorem 3.3.1 to construct solutions $U \in H^1(\Omega)$ for the magnetic Schrödinger operator $\mathcal{L}_{A,q}U = 0$ in Ω, with the required condition $U|_{\Gamma_0} = 0$. To achieve this condition we will use a reflection argument as in [27]. The main result of this section is Proposition 3.4.3.

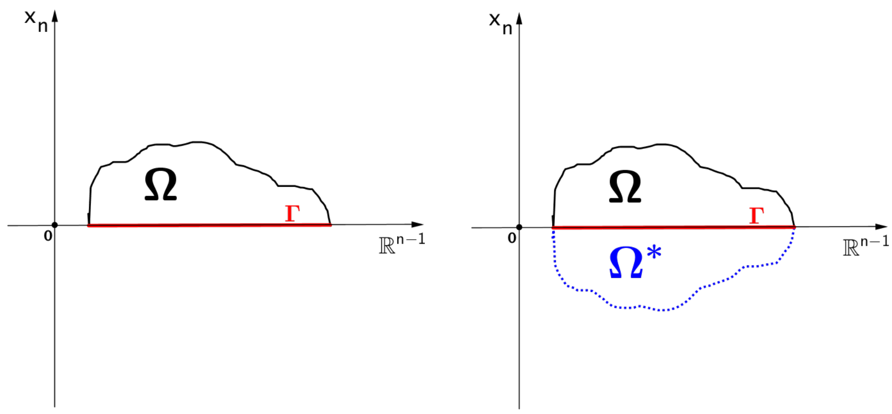

Figure 3.1: Description of the inaccessible part of the boundary Γ_0. Shape of Ω and Ω^*.

We set $x^* = (x', -x_n)$ for any $x = (x', x_n) \in \mathbb{R}^n$, $f^*(x) = f(x^*)$ for any function f and $E^* = \{x^* : x \in E\}$. Also for any $\rho \in \mathbb{C}^n$ we define $\rho^* = (\Re\rho)^* + i(\Im\rho)^*$. Then, similarly to

[27], we extend the magnetic an electric potentials from Ω to Ω^* by reflection with respect to the plane $\{x \in \mathbb{R}^n : x_n = 0\}$, see Figure 3.1. Recall that we have denoted by $A = (A^{(1)}, A^{(2)}, \ldots, A^{(n-1)}, A^{(n)})$ a magnetic potential. Then for $A^{(k)}$ with $k = 1, 2, \ldots, n-1$ we make an even extension and for $A^{(n)}$ we make an odd extension. We denote this extension by $\widetilde{A}$. More precisely, for all $k = 1, 2, \ldots, n-1$ we have:

$$\widetilde{A^{(k)}}(x) = \begin{cases} A^{(k)}(x', x_n), & x \in \Omega, \\ A^{(k)}(x', -x_n), & x \in \Omega^*, \end{cases}$$

and

$$\widetilde{A^{(n)}}(x) = \begin{cases} A^{(n)}(x', x_n), & x \in \Omega, \\ -A^{(n)}(x', -x_n), & x \in \Omega^*. \end{cases}$$

In the same way, for an electric potential q, we make an even extension. We denote these extensions by $\widetilde{q}$. More precisely, we have:

$$\widetilde{q_j}(x) = \begin{cases} q(x', x_n), & x \in \Omega \\ q(x', -x_n), & x \in \Omega^*. \end{cases}$$

The following lemma gives us the smoothness properties of these extensions.

Lemma 3.4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set. Let $M > 0$ and $s \in (0, 1/2)$. Consider $A \in L^\infty(\Omega; \mathbb{C}^n)$ and $q \in L^\infty(\Omega; \mathbb{C})$. If $\chi_\Omega A \in \mathcal{A}(\Omega, M, s)$ and $\chi_\Omega q \in \mathcal{Q}(\Omega, M, s)$ then $\chi_{\Omega \cup \Omega^*} \widetilde{A} \in \mathcal{A}(\Omega \cup \Omega^*, 2M, s)$ and $\chi_{\Omega \cup \Omega^*} \widetilde{q} \in \mathcal{Q}(\Omega \cup \Omega^*, 2M, s)$.*

Remark 3.4.2. *The sets $\mathcal{A}(\Omega, M, s)$ and $\mathcal{Q}(\Omega, M, s)$ denote the class of admissible magnetic and electric potentials. See definitions 1.2.2 and 1.2.3.*

Proof. The proof is based on the following observation. Let $y \in \mathbb{R}^n$ be fixed. If $f \in B_s^{2,\infty}(\mathbb{R}^n; \mathbb{C})$ then by Plancherel's theorem, we get

$$\|f(\cdot + y) - f(\cdot)\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left|\widehat{f}(\xi)\right|^2 \left|e^{-2\pi i \xi \cdot y} - 1\right|^2 d\xi,$$

which implies the following equivalent norm for the Besov spaces defined in (1.2.6):

$$\|f\|^2_{B_s^{2,\infty}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left|\widehat{f}(\xi)\right|^2 d\xi + \sup_{y \in \mathbb{R}^n \setminus \{0\}} \frac{\int_{\mathbb{R}^n} \left|\widehat{f}(\xi)\right|^2 \left|e^{-2\pi i \xi \cdot y} - 1\right|^2 d\xi}{|y|^{2s}}. \tag{3.4.1}$$

For the magnetic potential, according to the above identity, we first prove a relation between $\widehat{\chi_\Omega A}$ and $\widehat{\chi_{\Omega\cup\Omega^*}\widetilde{A}}$. For any $j = 1, 2, \ldots, n-1$, we have

$$\begin{aligned}
\widehat{\chi_{\Omega\cup\Omega^*}\widetilde{A^{(j)}}}(\xi) &= \int_{\mathbb{R}^n} e^{i\xi\cdot x}\chi_{\Omega\cup\Omega^*}\widetilde{A^{(j)}}dx = \int_{\Omega\cup\Omega^*} e^{i\xi\cdot x}\widetilde{A^{(j)}}dx \\
&= \int_\Omega e^{i\xi\cdot x}\widetilde{A^{(j)}}dx + \int_{\Omega^*} e^{i\xi\cdot x}\widetilde{A^{(j)}}dx \\
&= \int_\Omega e^{i\xi\cdot x}A^{(j)}(x)dx + \int_{\Omega^*} e^{i\xi\cdot x}A^{(j)}(x^*)dx \\
&= \int_\Omega e^{i\xi\cdot x}A^{(j)}(x)dx + \int_{\Omega} e^{i\xi^*\cdot x}A^{(j)}(x)dx \\
&= \widehat{\chi_\Omega A^{(j)}}(\xi) + \widehat{\chi_\Omega A^{(j)}}(\xi^*).
\end{aligned}$$

Hence, by (3.4.1) and the above identity, we get

$$\left\|\chi_{\Omega\cup\Omega^*}\widetilde{A^{(j)}}\right\|_{B_s^{2,\infty}(\mathbb{R}^n)} \leqslant 2\left\|\chi_\Omega A^{(j)}\right\|_{B_s^{2,\infty}(\mathbb{R}^n)} \leqslant 2M.$$

Analogously, for $j = n$, we obtain

$$\left\|\chi_{\Omega\cup\Omega^*}\widetilde{A^{(n)}}\right\|_{B_s^{2,\infty}(\mathbb{R}^n)} \leqslant 2\left\|\chi_\Omega A^{(n)}\right\|_{B_s^{2,\infty}(\mathbb{R}^n)} \leqslant 2M.$$

Moreover, since $A \in L^\infty(\Omega;\mathbb{C}^n)$ it follows that $\chi_{\Omega\cup\Omega^*}\widetilde{A} \in L^\infty(\mathbb{R}^n;\mathbb{C}^n)$. Thus, by combining this fact with the two above inequalities we obtain the desired result for A. The proof for q is analogous. So our proof is completed. $\square$

Now roughly we explain the main ideas to construct functions $U \in H^1(\Omega)$ satisfying $\mathcal{L}_{A,q}U = 0$ in Ω with the required condition on the inaccessible part of the boundary, that is $U|_{\Gamma_0} = 0$. At the beginning, one can apply Theorem 3.3.1 with $V = \Omega \cup \Omega^*$ in order to obtain $u \in H^1(\Omega \cup \Omega^*)$ satisfying $\mathcal{L}_{\chi_{\Omega\cup\Omega^*}\widetilde{A},\, \chi_{\Omega\cup\Omega^*}\widetilde{q}}u = 0$ in $\Omega \cup \Omega^*$. From the extensions of the magnetic and electric potentials It is easy to see that $U(x) := u(x) - u(x^*)$ is a solution of $\mathcal{L}_{A,q}U = 0$ in Ω. It only remains to prove the vanishing condition on Γ_0. It can be done by using an integration by parts but, a priori, this is not possible in Ω because we are not assuming any smoothness over $\partial\Omega$. To remedy this technical obstruction we will consider a ball B such that $\Omega \cup \Omega^* \subset\subset B$ and then we construct a solutions of the operator $\mathcal{L}_{A,q}U = 0$ in B^+, the upper half part of B. Since ∂B is now smooth, it is now possible to apply an integration by parts in B^+ in order to obtain $U|_{\partial B^+\cap\{x_n=0\}} = 0$. In particular, the restriction on Ω, $U|_\Omega$, satisfies $\mathcal{L}_{A,q}U|_\Omega = 0$. The vanishing condition on Γ_0 follows from $\Gamma_0 \subset \partial B^+ \cap \{x_n = 0\}$. The following proposition will be devoted to state and prove these ideas.

Proposition 3.4.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $A \in L^\infty(\Omega,\mathbb{C}^n)$ and $q \in L^\infty(\Omega,\mathbb{C})$. Given $M > 0$ and $s \in (0, 1/2)$, suposse that $\chi_\Omega A$ belongs to $\mathcal{A}(\Omega, M, s)$. Consider $\rho \in \mathbb{C}^n$ such that $\rho\cdot\rho = 0$ and $\rho = \rho_0 + \rho_\tau$ with ρ_0 being independent of some large parameter $\tau > 0$, $|\Re\rho_0| = |\Im\rho_0| = 1$ and $\rho_\tau = \mathcal{O}(\tau^{-1})$ as $\tau \mapsto \infty$. Then there exist* *two positive constants C and τ_0 (both depending on $n, \Omega, M, s, \|q\|_{L^\infty(\Omega)}$); and a solution $U \in H^1(\Omega)$ to the equation $\mathcal{L}_{A,q} U = 0$ in Ω with $U|_{\Gamma_0} = 0$ and of the form*

$$U(x,\rho;\tau) = u(x,\rho;\tau) - u(x^*,\rho;\tau), \quad x \in \Omega, \tag{3.4.2}$$

where $u \in H^1(\Omega \cup \Omega^*)$ is a function satisfying $\mathcal{L}_{\widetilde{A},\widetilde{q}} u = 0$ in $\Omega \cup \Omega^*$ and has the form:

$$u(x,\rho;\tau) = e^{\tau\rho\cdot x}\left(e^{\Phi^\sharp(x,\rho_0;\tau)} + r(x,\rho;\tau)\right), \quad x \in \Omega \cup \Omega^*. \tag{3.4.3}$$

Moreover we have the following properties:

(i) The function $\Phi^\sharp(\cdot,\rho_0;\tau) \in C^\infty(\mathbb{R}^n)$ and satisfies for all $\alpha \in \mathbb{N}^n$

$$\left\|\partial^\alpha \Phi^\sharp(\cdot,\rho_0;\tau)\right\|_{L^\infty(\mathbb{R}^n)} \leqslant C\tau^{|\alpha|/(s+2)}, \quad \tau \geqslant \tau_0. \tag{3.4.4}$$

(ii) The function $r(\cdot,\rho_0;\tau) \in H^1(\Omega \cup \Omega^*)$ and satisfies

$$\|\partial^\alpha r(\cdot,\rho_0;\tau)\|_{L^2(\Omega\cup\Omega^*)} \leqslant C\tau^{|\alpha|-s/(s+2)}, \quad |\alpha| \leqslant 1. \tag{3.4.5}$$

(iii) If we define by $\kappa := \sup_{x\in\overline{\Omega\cup\Omega^*}} |x|$ then the solution u satisfies

$$\|u\|_{H^1(\Omega\cup\Omega^*)} \leqslant C\, e^{\tau\kappa|\rho|}. \tag{3.4.6}$$

If we denote by $\Phi(\cdot;\rho_0) = (\rho_0 \cdot \nabla)^{-1}(-i\rho_0 \cdot (\chi_{\Omega\cup\Omega^*}\widetilde{A})) \in L^\infty(\mathbb{R}^n)$ the function satisfying the equation in $\mathbb{R}^n$

$$\rho_0 \cdot \nabla\Phi + i\rho_0 \cdot (\chi_{\Omega\cup\Omega^*}\widetilde{A}) = 0 \tag{3.4.7}$$

then

$$\|\Phi(\cdot;\rho_0)\|_{L^\infty(\mathbb{R}^n)} \leqslant C. \tag{3.4.8}$$

Finally, for every $\chi \in C_0^\infty(\mathbb{R}^n)$ we have

$$\left\|\chi(\Phi^\sharp(\cdot,\rho_0;\tau) - \Phi(\cdot;\rho_0))\right\|_{L^2(\mathbb{R}^n)} \leqslant C\tau^{-s/(s+2)}, \tag{3.4.9}$$

where the constant C also depends on χ.

Proof. The proof is an immediate consequence of Theorem 3.3.1. Let B be a ball centered at some fixed point on Γ_0 and such that $\overline{\Omega \cup \Omega^*} \subset B$. By hypothesis $\chi_\Omega A$ belongs to $\mathcal{A}(\Omega, M, s)$. Thus, by Lemma 3.4.1, we have that $\chi_{\Omega\cup\Omega^*}\widetilde{A} \in \mathcal{A}(\Omega\cup\Omega^*, 2M, s)$. Since the function $\chi_{\Omega\cup\Omega^*}\widetilde{A}$ is zero out of B, we deduce that $\chi_{\Omega\cup\Omega^*}\widetilde{A}$ also belongs to $\mathcal{A}(B, 2M, s)$, which imply that $\chi_{\Omega\cup\Omega^*}\widetilde{A} \in L^\infty \cap B_s^{2,\infty}(\mathbb{R}^n, \mathbb{C}^n)$ and $\mathrm{supp}(\chi_{\Omega\cup\Omega^*}\widetilde{A}) \subset B$. Notice also that $\chi_{\Omega\cup\Omega^*}\widetilde{q} \in L^\infty(\mathbb{R}^n, \mathbb{C})$ and $\mathrm{supp}(\chi_{\Omega\cup\Omega^*}\widetilde{q}) \subset B$. Then, by Theorem 3.3.1 applied to the functions $\chi_{\Omega\cup\Omega^*}\widetilde{A}$ and $\chi_{\Omega\cup\Omega^*}\widetilde{q}$ and $V = B$; there exist two positive constants C and τ_0 (both depending on $n, \Omega, M, \|q\|_{L^\infty(\Omega)}$); and a function $u \in H^1(B)$ of the form

$$u(x,\rho;\tau) = e^{\tau\rho\cdot x}\left(e^{\Phi^\sharp(x,\rho_0;\tau)} + r(x,\rho;\tau)\right), \quad \tau \geqslant \tau_0,$$

satisfying in B:

$$\mathcal{L}_{\chi_{\Omega\cup\Omega^*}\widetilde{A},\chi_{\Omega\cup\Omega^*}\widetilde{q}}u = 0 \tag{3.4.10}$$

with the corresponding estimates (3.3.1)-(3.3.6). These estimates imply the estimates (3.4.4)-(3.4.9). Now by a straightforward computation we have

$$\mathcal{L}_{\chi_\Omega A,\chi_\Omega q}u(x) = 0, \quad x \in B^+$$

and

$$\mathcal{L}_{\chi_\Omega A,\chi_\Omega q}u(x^*) = 0, \quad x \in B^+,$$

where B^+ denotes the upper half part of B, that is $B^+ = \{x \in B : x_n > 0\}$. Thus, from the two above equations we immediately deduce that the function defined by $U(x) := u(x) - u(x^*)$ satisfies $\mathcal{L}_{\chi_\Omega A,\chi_\Omega q}U = 0$ in B^+. Also, by integration be parts, it is easy to deduce that $U(x) = 0$ on $\partial B^+ \cap \{x_n = 0\}$. Finally, it is clear that U restricted to Ω, still denoted by U, satisfies the assertion of the proposition. The proof is completed. □

The next step will be to use Proposition 3.4.3 with some suitable ρ_1 and ρ_2 to construct functions $U_1, U_2 \in H^1(\Omega)$, for some suitable ρ_1 and ρ_2, satisfying $\mathcal{L}_{A_1,q_1}U_1 = 0$ with $U_1|_{\Gamma_0} = 0$ and $\mathcal{L}_{\overline{A_2},\overline{q_2}}U_2 = 0$ with $U_2|_{\Gamma_0} = 0$. Plugging these solutions into (3.2.1) we shall obtain information about $dA_1 - dA_2$.

Firstly, we shall give the motivation behind the choice of ρ_1 and ρ_2. Given $\xi \in \mathbb{R}^n$, let μ_1 and μ_2 be unit vectors in $\mathbb{R}^n$ such that

$$\xi\cdot\mu_1 = \xi\cdot\mu_2 = \mu_1\cdot\mu_2 = 0. \tag{3.4.11}$$

Then, as in [28], for a large parameter $\tau > 0$ we set

$$\begin{aligned} \rho_1 &= \frac{i}{2}\tau^{-1}\xi + i\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\mu_1 + \mu_2,\\ \rho_2 &= -\frac{i}{2}\tau^{-1}\xi + i\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\mu_1 - \mu_2. \end{aligned} \tag{3.4.12}$$

Observe that ρ_1 and ρ_2 can be written as follows:

$$\rho_1 = \rho_{1,0} + \mathcal{O}(\tau^{-1}),\ \rho_2 = \rho_{2,0} + \mathcal{O}(\tau^{-1}), \quad |\rho_1| = |\rho_2| = |\rho_1^*| = |\rho_2^*| = \sqrt{2}, \tag{3.4.13}$$

where

$$\rho_{1,0} = i\mu_1 + \mu_2, \quad \rho_{2,0} = i\mu_1 - \mu_2. \tag{3.4.14}$$

Now, by Proposition 3.4.3, for such ρ_1 there exists $U_1 \in H^1(\Omega)$ of the form

$$U_1 = e^{\tau\rho_1\cdot x}\left(e^{\Phi_1^\sharp} + r_1\right) - e^{\tau\rho_1^*\cdot x}\left(e^{\Phi_1^{\sharp^*}} + r_1^*\right)$$

satisfying $\mathcal{L}_{A_1,q_1}U_1 = 0$ and $U_1|_{\Gamma_0} = 0$. Analogously, for ρ_2 there exists $U_2 \in H^1(\Omega)$ of the form

$$U_2 = e^{\tau\rho_2\cdot x}\left(e^{\Phi_2^\sharp} + r_2\right) - e^{\tau\rho_2^*\cdot x}\left(e^{\Phi_2^{\sharp^*}} + r_2^*\right)$$

satisfying $\mathcal{L}_{\overline{A_2},\overline{q_2}}U_2 = 0$ and $U_2|_{\Gamma_0} = 0$. In order to exploit the information, about $A_1 - A_2$, encoded in the integral estimate (3.2.1), we have to compute $DU_1\overline{U_2} + U_1\overline{DU_2}$ and $U_1\overline{U_2}$. So we have to compute expressions of the form:

$$\begin{aligned}
&e^{\tau(\rho_1+\overline{\rho_2})\cdot x}f_1(x) = e^{i\tau\xi\cdot x}f_1(x),\\
&e^{\tau(\rho_1^*+\overline{\rho_2^*})\cdot x}f_2(x) = e^{i\tau\xi^*\cdot x}f_2(x),\\
&e^{\tau(\rho_1+\overline{\rho_2^*})\cdot x}f_3(x) = e^{\left(\frac{i}{2}(\xi+\xi^*)+i\sqrt{\tau^2-\frac{|\xi|^2}{4}}(\mu_1-\mu_1^*)+\tau(\mu_2-\mu_2^*)\right)\cdot x}f_3(x),\\
&e^{\tau(\rho_1^*+\overline{\rho_2})\cdot x}f_4(x) = e^{\left(\frac{i}{2}(\xi^*+\xi)-i\sqrt{\tau^2-\frac{|\xi|^2}{4}}(\mu_1-\mu_1^*)-\tau(\mu_2-\mu_2^*)\right)\cdot x}f_4(x),
\end{aligned}$$

for some suitable functions f_1, f_2, f_3 and f_4. We will se that the expressions involving f_1 and f_2 will give us information about the difference of the magnetic potentials $A_1 - A_2$. To estimate the expressions involving f_3 and f_4 we will use a quantitive version of the Riemann–Lebesgue lemma. Thus, we have to fix a priori μ_1 and μ_2 satisfying:

$$\mu_2 = \mu_2^*, \tag{3.4.15}$$

$$\lim_{\tau\to\infty}\left|\frac{1}{2}(\xi+\xi^*) + \sqrt{\tau^2 - \frac{|\xi|^2}{4}}(\mu_1-\mu_1^*)\right| = +\infty, \tag{3.4.16}$$

and

$$\lim_{\tau\to\infty}\left|\frac{1}{2}(\xi+\xi^*) - \sqrt{\tau^2 - \frac{|\xi|^2}{4}}(\mu_1-\mu_1^*)\right| = +\infty. \tag{3.4.17}$$

To fix such μ_1 and μ_2 satisfying (3.4.15)-(3.4.17) we proceed as in [25].

Given $\xi = (\xi_1, \xi_2, \dots, \xi_{n-1}, \xi_n) \in \mathbb{R}^n$, we denote $\xi' = (\xi_1, \xi_2, \dots, \xi_{n-1})$. Thus, we write $\xi = (\xi', \xi_n)$. Given $l = 1, 2, \dots, n-1$; we set

$$E_l := \left\{\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n : 0 < \sum_{\substack{k=1\\k\neq l}}^{n-1}\xi_k^2\right\}, \tag{3.4.18}$$

and for each $\xi \in \bigcap_{l=1}^{n-1} E_l$, we consider the following unit vectors in $\mathbb{R}^n$:

$$e(1) := \frac{1}{|\xi'|}(\xi', 0), \quad e(2), \quad e_n \tag{3.4.19}$$

with

$$e(2) \in (\text{span}\,\{e(1), e_n\})^{\perp} \quad \text{and} \quad e(2) = e(2)^*, \tag{3.4.20}$$

where e_n denotes the n-th canonical unit vector in $\mathbb{R}^n$. Notice that every $\xi \in \bigcap_{l=1}^{n-1} E_l$ can be written as $\xi = |\xi'|\, e(1) + \xi_n e_n$. The following lemma will be useful in the next computations.

Lemma 3.4.4. *For each $\xi \in \bigcap_{l=1}^{n-1} E_l$ and given $j, k = 1, \ldots, n$; there exist constants α, β and unit vectors μ_1, μ_2 satisfying (3.4.11), (3.4.15)-(3.4.17), such that*

$$\xi_j e_k - \xi_k e_j = \alpha\mu_1 + \beta\mu_2, \quad j, k = 1, 2, \ldots, n, \tag{3.4.21}$$

where e_l denotes the l-th canonical unit vector in $\mathbb{R}^n$. Moreover, μ_1 can be chosen independent of $j, k = 1, 2, \ldots, n$ and of the following form:

$$\mu_1 = -\frac{\xi_n}{|\xi|}e(1) + \frac{|\xi'|}{|\xi|}e_n. \tag{3.4.22}$$

Proof. Notice that for every $\xi \in \bigcap_{l=1}^{n-1} E_l$ we deduce that $|\xi'| > 0$ and then $|\xi| > 0$. So, μ_1 in (3.4.22) is well-defined. It is immediate to see that for $j, k = 1, 2, \ldots, n-1$; the unit vectors μ_1 and μ_2 defined by

$$\mu_1 := -\frac{\xi_n}{|\xi|}e(1) + \frac{|\xi'|}{|\xi|}e_n, \quad \mu_2 = (\mu_2)_{j,k} := (\xi_j e_k - \xi_k e_j)/\xi_j^2 + \xi_k^2 \tag{3.4.23}$$

satisfy (3.4.11) and (3.4.15)-(3.4.17). Moreover, we have the following identity

$$\xi_j e_k - \xi_k e_j = 0\mu_1 + (\xi_j^2 + \xi_k^2)\mu_2, \quad j, k = 1, 2, \ldots, n-1. \tag{3.4.24}$$

Here $\alpha = 0$ and $\beta = \xi_j^2 + \xi_k^2$. It remains to prove (3.4.21) for vectors of the form $\xi_j e_n - \xi_n e_j$ with $j = 1, 2, \ldots, n-1$. To prove that, we consider μ_1 as in (3.4.23), from which we deduce that

$$e_n = \frac{|\xi|}{|\xi'|}\mu_1 + \frac{\xi_n}{|\xi'|}e(1),$$

and then we would like to find two constants, α and β, and one unit vector μ_2 satisfying, together with μ_1, the conditions (3.4.11), (3.4.15)-(3.4.17); such that the following equality

$$\xi_j e_n - \xi_n e_j = \frac{\xi_j\,|\xi|}{|\xi'|}\mu_1 + \frac{\xi_j \xi_n}{|\xi'|}e(1) - \xi_n e_j = \alpha\mu_1 + \beta\mu_2, \quad j = 1, 2, \ldots, n-1,$$

holds true. Since μ_1 and μ_2 have to be orthogonal, from the above identity and a standard computation imply that

$$\alpha = \frac{\xi_j\,|\xi|}{|\xi'|}, \quad \beta = \frac{\xi_n}{|\xi'|}\left(|\xi'|^2 - \xi_j^2\right)^{1/2}$$

and

$$\mu_2 = (\mu_2)_{j,n} := \left(\left|\xi'\right|^2 - \xi_j^2\right)^{-1/2} \left(\xi_j e(1) - \left|\xi'\right| e_j\right), \quad j = 1, 2, \dots, n-1.$$

It is easy to check that such vectors satisfy, together with μ_1, the required conditions (3.4.11) and (3.4.15)-(3.4.17). Thus, the proof is completed. □

From now on, unless otherwise stated, we consider ρ_1 and ρ_2 as in (3.4.12) with μ_1 and μ_2 given by Lemma 3.4.4. Hence, we have the following equalities:

$$\begin{aligned} &\tau(\rho_1 + \overline{\rho_2}) \cdot x = i\xi \cdot x, \quad \tau(\rho_1^* + \overline{\rho_2^*}) \cdot x = i\xi^* \cdot x, \\ &\tau(\rho_1 + \overline{\rho_2^*}) \cdot x = i \left(\xi', 2\sqrt{\tau^2 - \frac{|\xi|^2}{4}} \frac{|\xi'|}{|\xi|}\right) \cdot x, \\ &\tau(\rho_1^* + \overline{\rho_2}) \cdot x = i \left(\xi', -2\sqrt{\tau^2 - \frac{|\xi|^2}{4}} \frac{|\xi'|}{|\xi|}\right) \cdot x. \end{aligned} \tag{3.4.25}$$

3.5 A Fourier estimate for the magnetic fields

This section will be devoted to proving the following proposition.

Proposition 3.5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $A_1, A_2 \in L^\infty(\Omega, \mathbb{C}^n)$ and $q_1, q_2 \in L^\infty(\Omega, \mathbb{C})$. Given $M > 0$ and $s \in (0, 1/2)$, assume that $\chi_\Omega A_1$ and $\chi_\Omega A_2$ belong to $\mathcal{A}(\Omega, M, s)$. Then there exist three positive constants C, τ_0 and ε_0 (all depending on $\Omega, n, M, s, \|q_1\|_{L^\infty}, \|q_2\|_{L^\infty}$) such that the following estimate:*

$$\begin{aligned} &\left|\mathcal{F}\left[d(\chi_{\Omega\cup\Omega^*}\widetilde{A_1})\right](\xi) - \mathcal{F}\left[d(\chi_{\Omega\cup\Omega^*}\widetilde{A_2})\right](\xi)\right| \\ &\leqslant C\,|\xi| \left[\tau^{-s/(s+2)} + e^{2\tau\kappa} dist\left(C_1^\Gamma, C_2^\Gamma\right) + \tau^{s/(s+2)}\left(e^{-4\pi\varepsilon^2\tau^2\frac{|\xi'|^2}{|\xi|^2}} + \varepsilon^s\right)\right], \end{aligned} \tag{3.5.1}$$

holds true for all $\xi \in \bigcap_{l=1}^{n-1} E_l$, $\tau \geqslant \tau_0$ and for all $0 < \varepsilon < \varepsilon_0$.

Remark 3.5.2. *We emphasize that the constant C is independent of ξ. To prove this proposition we will use two known results. The first one is a quantitative version of the Riemann–Lebesgue lemma. It was derived by Heck and Wang, see Lemma 2.1 in [24].*

Lemma 3.5.3. *Assume that $f \in L^1(\mathbb{R}^n)$ and there exist $\sigma > 0, C_0 > 0$, and $s \in (0, 1)$ such that*

$$\|f(\cdot + y) - f(\cdot)\|_{L^1(\mathbb{R}^n)} \leqslant C_0\,|y|^s \tag{3.5.2}$$

whenever $|y| < \sigma$. Then there exist two positive constants K and ε_0 such that for any $0 < \varepsilon < \varepsilon_0$, the inequality

$$\left|\widehat{f}(\xi)\right| \leqslant C_0 K (e^{-\pi\varepsilon^2|\xi|^2} + \varepsilon^s),$$

holds true with $K = K(\|f\|_{L^1}, n, \sigma, s)$.

The second result is a well-known result on nonlinear Fourier transform. For a proof see Proposition 3.3 in [28] and also Lemma 2.6 in [50].

Lemma 3.5.4. *Let $\xi, \mu_1, \mu_2 \in \mathbb{R}^n$ $(n \geqslant 3)$ be orthogonal vectors such that $|\mu_2| = |\mu_1| = 1$. If $W \in (L^\infty \cap \mathcal{E}')(\mathbb{R}^n; \mathbb{C}^n)$ and Φ satisfies*

$$(i\mu_1 + \mu_2) \cdot \nabla\Phi + (i\mu_1 + \mu_2) \cdot W = 0$$

in $\mathbb{R}^n$ then

$$(i\mu_1 + \mu_2) \cdot \int_{\mathbb{R}^n} W(x) e^{i\xi\cdot x} e^{\Phi(x)} dx = (i\mu_1 + \mu_2) \cdot \int_{\mathbb{R}^n} W(x) e^{i\xi\cdot x} dx.$$

We are now in the position to prove Proposition 3.5.1.

Proof. We shall start by computing the right-hand side of (3.2.1) multiplied by τ^{-1}, i.e. the task is now to estimate the expression

$$\tau^{-1} \int_\Omega \left[(A_1 - A_2) \cdot (DU_1\overline{U_2} + U_1\overline{DU_2}) + (A_1^2 - A_2^2 + q_1 - q_2)U_1\overline{U_2}\right] dx,$$

using the solutions $U_1, U_2 \in H^1(\Omega)$ given by Proposition 3.4.3. More precisely, for A_1, q_1 and ρ_1 given by (3.4.12), Proposition 3.4.3 ensures the existence of a function $U_1 \in H^1(\Omega)$ satisfying $\mathcal{L}_{A_1,q_1}U_1 = 0$ in Ω with $U_1|_{\Gamma_0} = 0$, having the form:

$$U_1(x) = e^{\tau\rho_1\cdot x}\left(e^{\Phi_1^\sharp} + r_1\right) - e^{\tau\rho_1^*\cdot x}\left(e^{\Phi_1^{\sharp*}} + r_1^*\right). \tag{3.5.3}$$

Analogously, by Proposition 3.4.3 now applied to $\overline{A_2}, \overline{q_2}$ and ρ_2 defined by (3.4.12), there exists $U_2 \in H^1(\Omega)$ satisfying $\mathcal{L}_{\overline{A_2},\overline{q_2}}U_2 = 0$ in Ω with $U_2|_{\Gamma_0} = 0$ having the form:

$$U_2(x) = e^{\tau\rho_2\cdot x}\left(e^{\Phi_2^\sharp} + r_2\right) - e^{\tau\rho_2^*\cdot x}\left(e^{\Phi_2^{\sharp*}} + r_2^*\right). \tag{3.5.4}$$

Both solutions have the following properties. The functions $\Phi_1^\sharp(\cdot, \rho_{0,1}; \tau)$ and $\Phi_2^\sharp(\cdot, \rho_{0,2}; \tau)$ belong to $C^\infty(\mathbb{R}^n)$ and satisfy for all $\alpha \in \mathbb{N}^n$

$$\left\|\partial^\alpha \Phi_i^\sharp\right\|_{L^\infty(\mathbb{R}^n)} + \left\|\partial^\alpha \Phi_i^{\sharp*}\right\|_{L^\infty(\mathbb{R}^n)} \leqslant C\tau^{|\alpha|/(s+2)}, \quad \tau \geqslant \tau_0,\ i = 1, 2. \tag{3.5.5}$$

For $i = 1, 2$, the functions r_i and r_i^* belong to $H^1(\Omega \cup \Omega^*)$ and satisfy

$$\|\partial^\alpha r_i\|_{L^2(\Omega\cup\Omega^*)} + \|\partial^\alpha r_i^*\|_{L^2(\Omega\cup\Omega^*)} \leqslant C\tau^{|\alpha|-s/(s+2)}, \quad |\alpha| \leqslant 1. \tag{3.5.6}$$

Moreover, from (3.4.13), we get

$$\|U_i\|_{H^1(\Omega)} \leqslant C\, e^{\tau\kappa|\rho|} \leqslant Ce^{\tau\kappa}, \quad i = 1, 2. \tag{3.5.7}$$

Also, from (3.4.7), the function $\Phi_1(\cdot;\rho_{1,0}) = (\rho_{1,0}\cdot\nabla)^{-1}(-i\rho_{1,0}\cdot(\chi_{\Omega\cup\Omega^*}\widetilde{A_1})) \in L^\infty(\mathbb{R}^n)$ satisfies the following equation in $\mathbb{R}^n$

$$\rho_{1,0}\cdot\nabla\Phi_1 + i\rho_{1,0}\cdot(\chi_{\Omega\cup\Omega^*}\widetilde{A_1}) = 0. \tag{3.5.8}$$

Also, the function $\Phi_2(\cdot;\rho_{2,0}) = (\rho_{2,0}\cdot\nabla)^{-1}(-i\rho_{2,0}\cdot\overline{(\chi_{\Omega\cup\Omega^*}\widetilde{A_2})}) \in L^\infty(\mathbb{R}^n)$, satisfies the equation in $\mathbb{R}^n$

$$\rho_{2,0}\cdot\nabla\Phi_2 + i\rho_{2,0}\cdot\overline{(\chi_{\Omega\cup\Omega}\widetilde{A_2})} = 0. \tag{3.5.9}$$

From (3.4.8), both functions satisfy the estimate

$$\|\Phi_i(\cdot;\rho_{i,0})\|_{L^\infty(\mathbb{R}^n)} \leqslant C, \quad i=1,2. \tag{3.5.10}$$

Finally, from (3.4.9), for every $\chi\in C_0^\infty(\mathbb{R}^n)$ we have

$$\left\|\chi(\Phi_i^\sharp(\cdot,\rho_0;\tau) - \Phi_i(\cdot;\rho_0))\right\|_{L^2(\mathbb{R}^n)} \leqslant C_1\tau^{-s/(s+2)}, \quad i=1,2. \tag{3.5.11}$$

With these solutions and properties at hand, and by a straightforward computation, we get

$$\begin{aligned}
&\tau^{-1}\int_\Omega (A_1-A_2)\cdot(DU_1\overline{U_2} + U_1\overline{DU_2})\\
&= i\int_\Omega(\overline{\rho_2}-\rho_1)\cdot(A_1-A_2)e^{i(\rho_1+\overline{\rho_2})\cdot x}e^{\Phi_1^\sharp+\overline{\Phi_2^\sharp}}\\
&+ i\int_\Omega(\overline{\rho_2^*}-\rho_1^*)\cdot(A_1-A_2)e^{i(\rho_1^*+\overline{\rho_2^*})\cdot x}e^{\Phi_1^{\sharp^*}+\overline{\Phi_2^{\sharp^*}}}\\
&+ i\int_\Omega(\rho_1-\overline{\rho_2^*})\cdot(A_1-A_2)e^{i(\ \rho_1+\overline{\rho_2^*})\cdot x}e^{\Phi_1^\sharp+\overline{\Phi_2^{\sharp^*}}}\\
&+ i\int_\Omega(\rho_1^*-\overline{\rho_2})\cdot(A_1-A_2)e^{i(\rho_1^*+\overline{\rho_2})\cdot x}e^{\Phi_1^{\sharp^*}+\overline{\Phi_2^\sharp}} + \int_\Omega R\cdot(A_1-A_2),
\end{aligned} \tag{3.5.12}$$

where R denotes the following expression:

$$\begin{aligned}
R &= i(\overline{\rho_2}-\rho_1)e^{\tau(\rho_1+\overline{\rho_2})\cdot x}(e^{\Phi_1^\sharp}\overline{r_2} + r_1\overline{e^{\Phi_2^\sharp}} + r_1\overline{r_2})\\
&+ i(\overline{\rho_2^*}-\rho_1^*)e^{\tau(\rho_1^*+\overline{\rho_2^*})\cdot x}(e^{\Phi_1^{\sharp^*}}\overline{r_2^*} + r_1^*\overline{e^{\Phi_2^{\sharp^*}}} + r_1^*\overline{r_2^*})\\
&+ i(\rho_1-\overline{\rho_2^*})e^{\tau(\rho_1+\overline{\rho_2^*})\cdot x}(e^{\Phi_1^\sharp}\overline{r_2^*} + r_1\overline{e^{\Phi_2^{\sharp^*}}} + r_1\overline{r_2^*})\\
&+ i(\rho_1^*-\overline{\rho_2})e^{\tau(\rho_1^*+\overline{\rho_2})\cdot x}(e^{\Phi_1^{\sharp^*}}\overline{r_2} + r_1^*\overline{e^{\Phi_2^\sharp}} + r_1^*\overline{r_2})\\
&+ i\tau^{-1}e^{\tau(\rho_1+\overline{\rho_2})\cdot x}\left[(e^{\Phi_1^\sharp}+r_1)\overline{\nabla(e^{\Phi_2^\sharp}+r_2)} - \overline{(e^{\Phi_2^\sharp}+r_2)}\nabla(e^{\Phi_1^\sharp}+r_1)\right]\\
&+ i\tau^{-1}e^{\tau(\rho_1^*+\overline{\rho_2^*})\cdot x}\left[(e^{\Phi_1^{\sharp^*}}+r_1^*)\overline{\nabla(e^{\Phi_2^{\sharp^*}}+r_2^*)} - \overline{(e^{\Phi_2^{\sharp^*}}+r_2^*)}\nabla(e^{\Phi_1^{\sharp^*}}+r_1^*)\right]\\
&+ i\tau^{-1}e^{\tau(\rho_1+\overline{\rho_2^*})\cdot x}\left[\overline{(e^{\Phi_2^{\sharp^*}}+r_2^*)}\nabla(e^{\Phi_1^\sharp}+r_1) - (e^{\Phi_1^\sharp}+r_1)\overline{\nabla(e^{\Phi_2^{\sharp^*}}+r_2^*)}\right]\\
&+ i\tau^{-1}e^{\tau(\rho_1^*+\overline{\rho_2})\cdot x}\left[\overline{(e^{\Phi_2^\sharp}+r_2)}\nabla(e^{\Phi_1^{\sharp^*}}+r_1^*) - (e^{\Phi_1^{\sharp^*}}+r_1^*)\overline{\nabla(e^{\Phi_2^\sharp}+r_2)}\right].
\end{aligned}$$

Since we have done an even extension for $A_i^{(j)}$ with $j = 1, 2, \dots, n-1$ and odd extension for $A_i^{(n)}$ for $i = 1, 2$; we have

$$\begin{aligned} & i\int_\Omega (\overline{\rho_2} - \rho_1)\cdot(A_1 - A_2)e^{i(\rho_1+\overline{\rho_2})\cdot x}e^{\Phi_1^\sharp + \overline{\Phi_2^\sharp}} \\ & + i\int_\Omega (\overline{\rho_2^*} - \rho_1^*)\cdot(A_1 - A_2)e^{i(\rho_1^*+\overline{\rho_2^*})\cdot x}e^{\Phi_1^{\sharp^*} + \overline{\Phi_2^{\sharp^*}}} \\ & = i\int_{\mathbb{R}^n} (\overline{\rho_2} - \rho_1)\cdot\left[\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2})\right]e^{i\xi\cdot x}e^{\Phi_1^\sharp + \overline{\Phi_2^\sharp}}. \end{aligned}$$

Replacing this equality, (3.4.12)-(3.4.14) and (3.4.25) into (3.5.12), we obtain

$$\begin{aligned} & \tau^{-1}\int_\Omega (A_1 - A_2)\cdot(DU_1\overline{U_2} + U_1\overline{DU_2}) \\ & = i\int_{\mathbb{R}^n} (\overline{\rho_2} - \rho_1)\cdot\left[\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2})\right]e^{i\xi\cdot x}e^{\Phi_1^\sharp + \overline{\Phi_2^\sharp}} + \int_\Omega R\cdot(A_1 - A_2) \\ & \quad + i\int_\Omega (\rho_1 - \overline{\rho_2^*})\cdot(A_1 - A_2)e^{i\left(\xi', 2\tau\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)\cdot x}e^{\Phi_1^\sharp + \overline{\Phi_2^{\sharp^*}}} \\ & \quad + i\int_\Omega (\rho_1^* - \overline{\rho_2})\cdot(A_1 - A_2)e^{i\left(\xi', -2\tau\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)\cdot x}e^{\Phi_1^{\sharp^*} + \overline{\Phi_2^\sharp}} \\ & = i\int_{\mathbb{R}^n} (\overline{\rho_{2,0}} - \rho_{1,0})\cdot\left[\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2})\right]e^{i\xi\cdot x}e^{\Phi_1^\sharp + \overline{\Phi_2^\sharp}} \\ & \quad + \int_{\mathbb{R}^n} \mathcal{O}(\tau^{-1})\cdot\left[\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2})\right]e^{i\xi\cdot x}e^{\Phi_1^\sharp + \overline{\Phi_2^\sharp}} + \int_\Omega R\cdot(A_1 - A_2) \\ & \quad + i\int_{\mathbb{R}^n} (\rho_1 - \overline{\rho_2^*})\cdot(\chi_\Omega(A_1 - A_2))e^{i\left(\xi', 2\tau\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)\cdot x}e^{\Phi_1^\sharp + \overline{\Phi_2^{\sharp^*}}} \\ & \quad + i\int_{\mathbb{R}^n} (\rho_1^* - \overline{\rho_2})\cdot(\chi_\Omega(A_1 - A_2))e^{i\left(\xi', -2\tau\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)\cdot x}e^{\Phi_1^{\sharp^*} + \overline{\Phi_2^\sharp}}. \end{aligned}$$

Hence, from this identity and also adding and subtracting terms, we get

$$\begin{aligned} & i\int_{\mathbb{R}^n} (\overline{\rho_{2,0}} - \rho_{1,0})\cdot\left[\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2})\right]e^{i\xi\cdot x}e^{\Phi_1 + \overline{\Phi_2}} \\ & \qquad = I + II + III + IV + V + VI + VII, \end{aligned} \tag{3.5.13}$$

where

$$I = i\int_{\mathbb{R}^n} (\overline{\rho_{2,0}} - \rho_{1,0})\cdot\left[\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2})\right]e^{i\xi\cdot x}(e^{\Phi_1 + \overline{\Phi_2}} - e^{\Phi_1^\sharp + \overline{\Phi_2^\sharp}}), \tag{3.5.14}$$

$$\begin{aligned} II = \tau^{-1}\int_\Omega & \left[(A_1 - A_2)\cdot(DU_1\overline{U_2} + U_1\overline{DU_2})\right. \\ & \left. + (A_1^2 - A_2^2 + q_1 - q_2)U_1\overline{U_2}\right], \end{aligned} \tag{3.5.15}$$

$$III = -\tau^{-1} \int_\Omega (A_1^2 - A_2^2 + q_1 - q_2) U_1 \overline{U_2}, \tag{3.5.16}$$

$$IV = -\int_{\mathbb{R}^n} \mathcal{O}(\tau^{-1}) \cdot \left[\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2})\right] e^{i\xi\cdot x} e^{\Phi_1^\sharp + \overline{\Phi_2^\sharp}}, \tag{3.5.17}$$

$$V = \int_\Omega R \cdot (A_1 - A_2), \tag{3.5.18}$$

$$VI = -i \int_{\mathbb{R}^n} (\rho_1 - \overline{\rho_2^*}) \cdot (\chi_\Omega(A_1 - A_2)) e^{i\left(\xi', 2\tau\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)\cdot x} e^{\Phi_1^\sharp + \overline{\Phi_2^{\sharp *}}}, \tag{3.5.19}$$

$$VII = -i \int_{\mathbb{R}^n} (\rho_1^* - \overline{\rho_2}) \cdot (\chi_\Omega(A_1 - A_2)) e^{i\left(\xi', -2\tau\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)\cdot x} e^{\Phi_1^{\sharp *} + \overline{\Phi_2^\sharp}}. \tag{3.5.20}$$

The task is now to estimate each one of the above terms. To estimate the first term I, we will use the following fact:

$$|e^{z_1} - e^{z_2}| \leqslant |z_1 - z_2|\, e^{\max\{\Re z_1, \Re z_2\}}, \tag{3.5.21}$$

for all $z_1, z_2 \in \mathbb{C}$. Thus, from (3.4.14), the boundedness of $\Omega \cup \Omega^*$ and (3.5.11), we obtain

$$|I| \leqslant C_1 \left\| \Phi_1 - \Phi_1^{\sharp *} + \overline{\Phi_2} - \Phi_2^{\sharp *} \right\|_{L^2(\mathbb{R}^n)} \leqslant C_2\, \tau^{-s/(s+2)}. \tag{3.5.22}$$

From Corollary 3.2.2 and from (3.5.7), we obtain

$$\begin{aligned} |II| &\leqslant C_3 \tau^{-1} dist\,(C_1^\Gamma, C_2^\Gamma)\, \|U_1\|_{H^1(\Omega)} \|U_2\|_{H^1(\Omega)} \\ &\leqslant C_4 \tau^{-1} e^{2\tau\kappa} dist\,(C_1^\Gamma, C_2^\Gamma). \end{aligned} \tag{3.5.23}$$

We continue in this fashion to estimate the other terms. The identities (3.5.3)-(3.5.4), (3.4.12), (3.4.23), the estimates (3.5.5)-(3.5.6) and the triangular inequality, imply that

$$
\begin{aligned}
|III| &= \tau^{-1}\left|\int_\Omega (A_1^2 - A_2^2 + q_1 - q_2) e^{\tau\overline{\rho_2}\cdot x} U_1 \overline{(e^{-\tau\rho_2\cdot x} U_2)}\right| \\
&\leqslant C_6\tau^{-1}\left\|e^{\tau\overline{\rho_2}\cdot x}U_1\right\|_{L^2(\Omega)}\left\|e^{-\tau\rho_2\cdot x}U_2\right\|_{L^2(\Omega)} \\
&= C_6\tau^{-1}\left\|e^{\tau\overline{\rho_2}\cdot x}\left[e^{\tau\rho_1\cdot x}\left(e^{\Phi_1^\sharp}+r_1\right) - e^{\tau\rho_1^*\cdot x}\left(e^{\Phi_1^{\sharp^*}}+r_1^*\right)\right]\right\|_{L^2(\Omega)} \\
&\times \left\|e^{-\tau\rho_2\cdot x}\left[e^{\tau\rho_2\cdot x}(e^{\Phi_2^\sharp}+r_2) - e^{\tau\rho_2^*\cdot x}\left(e^{\Phi_2^{\sharp^*}}+r_2^*\right)\right]\right\|_{L^2(\Omega)} \\
&= C_6\tau^{-1}\left\|e^{i\xi\cdot x}\left(e^{\Phi_1^\sharp}+r_1\right) - e^{i\xi^*\cdot x}\left(e^{\Phi_1^{\sharp^*}}+r_1^*\right)\right\|_{L^2(\Omega)} \\
&\times \left\|e^{\Phi_2^\sharp}+r_2 - e^{\tau(\rho_2^*-\rho_2)\cdot x}\left(e^{\Phi_2^{\sharp^*}}+r_2^*\right)\right\|_{L^2(\Omega)} \\
&\leqslant C_6\tau^{-1}\Bigg(\left\|e^{\Phi_1^\sharp}+r_1\right\|_{L^2(\Omega)} + \left\|e^{\Phi_1^{\sharp^*}}+r_1^*\right\|_{L^2(\Omega)} \\
&\qquad + \left\|e^{\Phi_2^\sharp}+r_2\right\|_{L^2(\Omega)} + \left\|e^{\Phi_2^{\sharp^*}}+r_2^*\right\|_{L^2(\Omega)}\Bigg) \leqslant C_7\,\tau^{-1},
\end{aligned}
\tag{3.5.24}
$$

where at the last line we have used the identity

$$
\rho_2^* - \rho_2 = i\left(-\frac{1}{2}(\xi+\xi^*) - 2\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}e(n)\right).
$$

Again, from (3.5.5) and the boundedness of $\Omega\cup\Omega^*$, it follows easily that

$$
|IV| = \left|-\int_{\mathbb{R}^n}\mathcal{O}(\tau^{-1})\cdot\left[\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2})\right]e^{i\xi\cdot x}e^{\Phi_1^\sharp+\overline{\Phi_2^\sharp}}\right| \leqslant C_8\,\tau^{-1}. \tag{3.5.25}
$$

From (3.4.12), (3.4.25) and (3.5.5)-(3.5.6), we get

$$
|V| = \left|\int_\Omega R\cdot(A_1-A_2)\right| \leqslant C_{10}\,\tau^{-s/(s+2)}. \tag{3.5.26}
$$

To estimate the term VI, we will use an extra argument due to Heck and Wang [24]. From (3.4.13), we have

$$
\begin{aligned}
|VI| &\leqslant \left|\rho_1-\overline{\rho_2^*}\right|\left|\int_{\mathbb{R}^n}(\chi_\Omega(A_1-A_2))e^{i\left(\xi',2\tau\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)\cdot x}e^{\Phi_1^\sharp+\overline{\Phi_2^{\sharp^*}}}\right| \\
&\leqslant 2\sqrt{2}\left|\mathcal{F}\left[\chi_\Omega(A_1-A_2)e^{\Phi_1^\sharp+\overline{\Phi_2^{\sharp^*}}}\right]\left(\xi',2\tau\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)\right|.
\end{aligned}
\tag{3.5.27}
$$

Now we would like to applied Lemma 3.5.3 to estimate the above expression. We claim that the function $\chi_\Omega(A_1-A_2)e^{\Phi_1^\sharp+\overline{\Phi_2^{\sharp *}}}$ satisfies the condition (3.5.2) of Lemma 3.5.3 with $\sigma=1$. Indeed, if we denote by $\Phi^\sharp=\Phi_1^\sharp+\overline{\Phi_2^{\sharp *}}$ then from (3.5.5) and by a standard interpolation between the spaces $C(\mathbb{R}^n)$ and $C^1(\mathbb{R}^n)$ we get

$$\left\|\Phi^\sharp\right\|_{C^s(\mathbb{R}^n)} \leqslant C_{10}\left\|\Phi^\sharp\right\|_{C(\mathbb{R}^n)}^{1-s}\left\|\Phi^\sharp\right\|_{C^1(\mathbb{R}^n)}^{s} \leqslant C_{11}\,\tau^{s/(s+2)}. \tag{3.5.28}$$

For convenience, we denote $A:=\chi_\Omega(A_1-A_2)$. Thus for any $y\in\mathbb{R}^n$, Cauchy-Schwarz inequality and (3.5.21) imply that

$$\begin{aligned}
&\left\|\left[\chi_\Omega(A_1-A_2)e^{\Phi_1^\sharp+\overline{\Phi_2^{\sharp *}}}\right](\cdot+y)-\left[\chi_\Omega(A_1-A_2)e^{\Phi_1^\sharp+\overline{\Phi_2^{\sharp *}}}\right](\cdot)\right\|_{L^1(\mathbb{R}^n)}\\
&=\left\|(Ae^{\Phi^\sharp})(\cdot+y)-(Ae^{\Phi^\sharp})(\cdot)\right\|_{L^1(\mathbb{R}^n)}\\
&=\int_{\mathbb{R}^n}\left|\left[A(x+y)-A(x)\right]e^{\Phi^\sharp(x+y)}+\left[e^{\Phi^\sharp(x+y)}-e^{\Phi^\sharp(x)}\right]A(x)\right|dx\\
&\leqslant\int_{\mathbb{R}^n}\left|\left[A(x+y)-A(x)\right]e^{\Phi^\sharp(x+y)}\right|dx+\int_{\mathbb{R}^n}\left|\left[e^{\Phi^\sharp(x+y)}-e^{\Phi^\sharp(x)}\right]A(x)\right|dx\\
&\leqslant C_{13}\left(\|A(\cdot+y)-A(\cdot)\|_{L^2(\mathbb{R}^n)}+\left\|\Phi^\sharp(\cdot+y)-\Phi^\sharp(\cdot)\right\|_{L^\infty(\mathbb{R}^n)}\right)\\
&\leqslant C_{14}\left(\|A\|_{B_s^{2,\infty}}+\left\|\Phi^\sharp\right\|_{C^s(\mathbb{R}^n)}\right)|y|^s.
\end{aligned}$$

Then, by combining the above inequality with (3.5.28), we obtain

$$\left\|(Ae^{\Phi^\sharp})(\cdot+y)-(Ae^{\Phi^\sharp})(\cdot)\right\|_{L^1(\mathbb{R}^n)}\leqslant C_{15}\,\tau^{s/(s+2)}\,|y|^s,$$

for any $y\in\mathbb{R}^n$ with $|y|<1$. So the claim is proved.

Hence, applying Lemma 3.5.3 to $f=\chi_\Omega(A_1-A_2)e^{\Phi_1^\sharp+\overline{\Phi_2^{\sharp *}}}$, $C_0=C_{15}\tau^{s/(s+2)}$ and $\sigma=1$, there exist $C_{15}>0$ and $\varepsilon_0>0$ such that the following inequality

$$\left|\mathcal{F}\left[\chi_\Omega(A_1-A_2)e^{\Phi_1^\sharp+\overline{\Phi_2^{\sharp *}}}\right](\eta)\right|\leqslant C_{15}\,\tau^{s/(s+2)}\left(e^{-\pi\varepsilon^2|\eta|^2}+\varepsilon^s\right),$$

holds true for all $0<\varepsilon<\varepsilon_0$ and for all $\eta\in\mathbb{R}^n$. Hence, from (3.5.27) and the above inequality applied to

$$\eta=\left(\xi',2\tau\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right),$$

we get

$$|VI|\leqslant C_{16}\,\tau^{s/(s+2)}\left(e^{-4\pi\varepsilon^2\tau^2\frac{|\xi'|^2}{|\xi|^2}}+\varepsilon^s\right). \tag{3.5.29}$$

Analogously as was estimated VI, we obtain

$$|VII| \leqslant C_{17}\,\tau^{s/(s+2)}\left(e^{-4\pi\varepsilon^2\tau^2\frac{|\xi'|^2}{|\xi|^2}}+\varepsilon^s\right). \tag{3.5.30}$$

Hence, by combining (3.5.22)-(3.5.26) and (3.5.29)-(3.5.30) into (3.5.13) and taking into account that $\overline{\rho_{2,0}}-\rho_{1,0}=-2(i\mu_1+\mu_2)$; we get

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}(i\mu_1+\mu_2)\cdot(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2}))e^{i\xi\cdot x}e^{\Phi_1+\overline{\Phi_2}}\right|\\&=\frac{1}{2}\left|\int_{\mathbb{R}^n}(\overline{\rho_{2,0}}-\rho_{1,0})\cdot(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2}))e^{i\xi\cdot x}e^{\Phi_1+\overline{\Phi_2}}\right|\\&\leqslant C_{18}\left[\tau^{-s/(s+2)}+e^{2\tau\kappa}dist\,(C_1^\Gamma,C_2^\Gamma)+\tau^{s/(s+2)}\left(e^{-4\pi\varepsilon^2\tau^2\frac{|\xi'|^2}{|\xi|^2}}+\varepsilon^s\right)\right].\end{aligned} \tag{3.5.31}$$

Now the next task will be to remove the function $e^{\Phi_1+\overline{\Phi_2}}$ from the left-hand side of the above inequality. Since (3.5.8) and (3.5.9), it follow easily that

$$(\overline{\rho_{2,0}}-\rho_{1,0})\cdot\nabla(\Phi_1+\overline{\Phi_2})-i(\overline{\rho_{2,0}}-\rho_{1,0})\cdot(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2}))=0,$$

which implies that

$$(i\mu_1+\mu_2)\cdot\nabla(\Phi_1+\overline{\Phi_2})-i(i\mu_1+\mu_2)\cdot(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2}))=0. \tag{3.5.32}$$

From this equation, (3.4.11), (3.5.10) and applying Lemma 3.5.4 to $\Phi=\Phi_1+\overline{\Phi_2}$ and $W=-i\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2})$, we can remove the term $e^{\Phi_1+\overline{\Phi_2}}$ from the left-hand side of (3.5.31). Thus, we obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}(i\mu_1+\mu_2)\cdot(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2}))e^{i\xi\cdot x}\right|\\&\leqslant C_{19}\left[\tau^{-s/(s+2)}+e^{2\tau k}dist\,(C_1^\Gamma,C_2^\Gamma)+\tau^{s/(s+2)}\left(e^{-4\pi\varepsilon^2\tau^2\frac{|\xi'|^2}{|\xi|^2}}+\varepsilon^s\right)\right].\end{aligned} \tag{3.5.33}$$

We can apply the previous arguments with ρ_1 replaced by $\overline{\rho_1}$ and ρ_2 replaced by $\overline{\rho_2}$, to obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}(-i\mu_1+\mu_2)\cdot(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2}))e^{i\xi\cdot x}\right|\\&\leqslant C_{20}\left[\tau^{-s/(s+2)}+e^{2\tau\kappa}dist\,(C_1^\Gamma,C_2^\Gamma)+\tau^{s/(s+2)}\left(e^{-4\pi\varepsilon^2\tau^2\frac{|\xi'|^2}{|\xi|^2}}+\varepsilon^s\right)\right].\end{aligned} \tag{3.5.34}$$

Thus, adding and subtracting (3.5.33) and (3.5.34), we get

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\mu\cdot(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2}))e^{i\xi\cdot x}\right|\\&\leqslant C_{21}\,|\mu|\left[\tau^{-s/(s+2)}+e^{2\tau\kappa}dist\,(C_1^\Gamma,C_2^\Gamma)+\tau^{s/(s+2)}\left(e^{-4\pi\varepsilon^2\tau^2\frac{|\xi'|^2}{|\xi|^2}}+\varepsilon^s\right)\right].\end{aligned} \tag{3.5.35}$$

for all $\mu \in \operatorname{span}\{\mu_1, \mu_2\}$ and for all $\xi \in \bigcap_{l=1}^{n-1} E_l$. Now Lemma 3.4.4 ensures that for every $j, k = 1, 2, \dots, n$ the vector defined by $(\mu)_{j,k} := \xi_j e_k - \xi_k e_j$ belongs to $\operatorname{span}\{\mu_1, \mu_2\}$. Thus, replacing these vectors into (3.5.35), it immediately implies the desired assertion of proposition. So the proof is completed. □

3.6 Proof of log-estimate for the magnetic fields

By Proposition 3.5.1, taking into account that the constant $C > 0$ in the estimate (3.5.1) is independent of $\xi \in \bigcap_{l=1}^{n-1} E_l$ and since the set $\bigcap_{l=1}^{n-1} E_l$ is dense in $\mathbb{R}^n$, it follows that the following estimate

$$\begin{aligned} &\left| \mathcal{F}\left[d(\chi_{\Omega\cup\Omega^*}\widetilde{A_1}) - \right](\xi) - \mathcal{F}\left[d(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}) \right](\xi) \right| \\ &\leqslant C\,|\xi| \left[\tau^{-s/(s+2)} + e^{2\tau\kappa} dist\,(C_1^\Gamma, C_2^\Gamma) + \tau^{s/(s+2)} \left(e^{-4\pi\varepsilon^2\tau^2 \frac{|\xi'|^2}{|\xi|^2}} + \varepsilon^s \right) \right], \end{aligned} \tag{3.6.1}$$

holds true for all $\xi \in \mathbb{R}^n$. Now consider $R \geqslant 1$ (which will be fixed later) and denote by $B_R(0)$ the open ball in $\mathbb{R}^n$ centered at 0 of radius R. For convenience we denote $\widetilde{A} := \chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2})$. Then

$$\left\| d\widetilde{A} \right\|^2_{H^{-1}(\mathbb{R}^n)} = \left\| d\widetilde{A} \right\|^2_{H^{-1}(\mathbb{R}^n)} = E_1 + E_2, \tag{3.6.2}$$

where

$$E_1 = \int_{B_R(0)\setminus\{0\}} (1 + |\xi|^2)^{-1} \left| \mathcal{F}\left[d\widetilde{A} \right](\xi) \right|^2 d\xi$$

and

$$E_2 = \int_{\mathbb{R}^n\setminus B_R(0)} (1 + |\xi|^2)^{-1} \left| \mathcal{F}\left[d\widetilde{A} \right](\xi) \right|^2 d\xi.$$

From (3.6.1) and taking $\varepsilon = \tau^{-3/2(s+2)}$, we get

$$\begin{aligned} E_1 &= \int_{B_R(0)\setminus\{0\}} (1 + |\xi|^2)^{-1} \left| \mathcal{F}\left[d\widetilde{A} \right](\xi) \right|^2 d\xi \\ &\leqslant C_1\, R^n \left(\tau^{-2s/(s+2)} + e^{4\tau\kappa} dist\,(C_1^\Gamma, C_2^\Gamma)^2 + \tau^{2s/(s+2)} \varepsilon^{2s} \right) \\ &\quad + C_1\, \tau^{2s/(s+2)} \int_{B_R(0)\setminus\{0\}} e^{-8\pi\varepsilon^2\tau^2 \frac{|\xi'|^2}{|\xi|^2}} \\ &\leqslant C_1\, R^n \left(\tau^{-2s/(s+2)} + e^{4\tau\kappa} dist\,(C_1^\Gamma, C_2^\Gamma)^2 + \tau^{2s/(s+2)} \varepsilon^{2s} \right) \\ &\quad + C_1 \tau^{2s/(s+2)} R^n \varepsilon^{-2} \tau^{-2} \\ &\leqslant C_2\, R^n \left(\tau^{-s/(s+2)} + e^{4\tau\kappa} dist\,(C_1^\Gamma, C_2^\Gamma)^2 \right). \end{aligned} \tag{3.6.3}$$

To estimate the integral over $\mathbb{R}^n \setminus B_R(0)$ in (3.6.2) we shall use a mollification argument to $\widetilde{A}$, as was done in the proof of Theorem 3.3.1. By Lemma 3.4.1, we have $\widetilde{A} \in L^\infty \cap$

$B_s^{2,\infty}(\mathbb{R}^n)$. Thus, consider $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leqslant \varphi \leqslant 1$ and $\operatorname{supp} \varphi \subset \overline{B_1(0)}$, where $\overline{B_1(0)}$ denotes the closure of the ball in $\mathbb{R}^n$ of radius 1 centered at the origin. For each $\delta > 0$ we define $\varphi_\delta(x) = \delta^{-n}\varphi(x/\delta)$ and set $\widetilde{A}_\delta^\sharp = \widetilde{A} * \varphi_\delta$ which belongs to $C_0^\infty(\mathbb{R}^n; \mathbb{C}^n)$. Then there exists a positive constant $C_3 > 0$ (depending on Ω and n) such that

$$\left\| \widetilde{A} - \widetilde{A}_\delta^\sharp \right\|_{L^2(\mathbb{R}^n)} \leqslant C_3\, \delta^s \left\| \widetilde{A} \right\|_{B_s^{2,\infty}(\mathbb{R}^n)}, \tag{3.6.4}$$

and for each $\alpha \in \mathbb{N}^n$, we have

$$\left\| \partial^\alpha \widetilde{A}_\delta^\sharp \right\|_{L^\infty(\mathbb{R}^n)} \leqslant C_3\, \delta^{-|\alpha|} \left\| \widetilde{A} \right\|_{L^\infty(\mathbb{R}^n)}. \tag{3.6.5}$$

Now, Plancherel's identity and (3.6.4)-(3.6.5), imply that

$$\begin{aligned}
E_2 &= \int_{\mathbb{R}^n \setminus B_R(0)} (1+|\xi|^2)^{-1} \left| \mathcal{F}\left[d\widetilde{A} \right](\xi) \right|^2 d\xi \\
&\leqslant C_4 \int_{\mathbb{R}^n \setminus B_R(0)} (1+|\xi|^2)^{-1} \left| \mathcal{F}\left[d\widetilde{A}_\delta^\sharp \right](\xi) \right|^2 d\xi \\
&\quad + C_4 \int_{\mathbb{R}^n \setminus B_R(0)} (1+|\xi|^2)^{-1} \left| \mathcal{F}\left[d\widetilde{A} \right](\xi) - \mathcal{F}\left[d\widetilde{A}_\delta^\sharp \right](\xi) \right|^2 d\xi \\
&\leqslant C_5 \int_{\mathbb{R}^n \setminus B_R(0)} (1+|\xi|^2)^{-1} \left| \mathcal{F}\left[d\widetilde{A}_\delta^\sharp \right](\xi) \right|^2 d\xi \\
&\quad + C_5 \int_{\mathbb{R}^n \setminus B_R(0)} |\xi|^2 (1+|\xi|^2)^{-1} \left| \mathcal{F}\left[A - A^\sharp \right](\xi) \right|^2 d\xi \\
&\leqslant C_6 \left(R^{-2} \left\| dA^\sharp \right\|_{L^2(\mathbb{R}^n)}^2 + \left\| \widetilde{A} - \widetilde{A}_\delta^\sharp \right\|_{L^2(\mathbb{R}^n)}^2 \right) \\
&\leqslant C_7 \left(R^{-2}\delta^{-2} + \delta^{2s} \right).
\end{aligned}$$

By equating the terms $R^{-2}\delta^{-2}$ and δ^{2s}, that is taking $\delta = R^{-1/(s+1)}$, we get

$$\int_{\mathbb{R}^n \setminus B_R(0)} (1+|\xi|^2)^{-1} \left| \mathcal{F}\left[d\widetilde{A} \right](\xi) \right|^2 d\xi \leqslant C_8 R^{-2s/(s+1)}. \tag{3.6.6}$$

Then combining (3.7.54) and (3.6.6) into (3.6.2), we have that there exist two positive constants C_9 and τ_1 such that the estimate

$$\begin{aligned}
&\left\| d\widetilde{A} \right\|_{H^{-1}(\mathbb{R}^n)}^2 \\
&\quad \leqslant C_9 \left(R^n e^{4\tau\kappa} dist\,(C_1^\Gamma, C_2^\Gamma)^2 + R^n \tau^{-s/(s+2)} + R^{-2s/(s+1)} \right),
\end{aligned} \tag{3.6.7}$$

holds true for all $\tau \geqslant \tau_1$. By equate the two last terms of the right-hand side of the above inequality, and then we take

$$R = \tau^{\frac{s(s+1)}{(s+2)(n+ns+2s)}}.$$

We have that there exist another two positive constants C_{10} and τ_2 such that

$$R^n = \tau^{\frac{ns(s+1)}{(s+2)(n+ns+2s)}} \leqslant C_{10} e^{\tau k},$$

for all $\tau \geqslant \tau_2$. Thus, replacing these facts into (3.6.7), we have

$$\begin{aligned} &\left\| d\widetilde{A} \right\|^2_{H^{-1}(\mathbb{R}^n)} \\ &\leqslant C_{11} \left(e^{5\tau\kappa} dist\,(C_1^\Gamma, C_2^\Gamma)^2 + \tau^{-\frac{2s^2}{(s+2)(n+ns+2s)}} \right). \end{aligned} \tag{3.6.8}$$

On the other hand, there exists $\tau_3 > 0$ such that

$$e^{-\tau\kappa} \leqslant \tau^{-\frac{2s^2}{(s+2)(n+ns+2s)}},$$

for all $\tau \geqslant \tau_3$. Now taking $\tau_0 \geqslant \max(\tau_1, \tau_2, \tau_3)$ such that $3\kappa\tau_0 \geqslant 1$, it is easy to check that

$$\tau := \frac{1}{3k} \left| \log dist(C_1^\Gamma, C_2^\Gamma) \right| \geqslant \tau_0, \tag{3.6.9}$$

whenever

$$dist(C_1^\Gamma, C_2^\Gamma) \leqslant e^{-3\kappa\tau_0}.$$

Thus, from (3.6.9) it follows that $dist(C_1^\Gamma, C_2^\Gamma) \leqslant e^{-3\tau k}$ and then from (3.6.8) we get

$$\left\| d\widetilde{A} \right\|_{H^{-1}(\mathbb{R}^n)} \leqslant C_{10} \left| \log dist(C_1^\Gamma, C_2^\Gamma) \right|^{-\frac{s^2}{(s+2)(n+ns+2s)}}. \tag{3.6.10}$$

Since $s \in (0, 1/2)$, it is immediate to see that

$$\frac{s^2}{(s+2)(n+ns+2s)} \geqslant \frac{s^2}{5n}.$$

Thus, by considering the above inequality into (3.6.10), taking $C = \max\{3\kappa\tau_0, C_{10}\}$, $\lambda = 1/5$ and finally taking into account that

$$\|d(A_1 - A_2)\|_{H^{-1}(\Omega)} \leqslant \left\| d\widetilde{A} \right\|_{H^{-1}(\mathbb{R}^n)}.,$$

we conclude the proof.

3.7 Proof of log-estimate for the electric potentials

The goal of this section is to prove Theorem 1.2.5. The idea will be to combine the gauge invariance of the Cauchy data sets, see Lemma 3.1 in [28], and the stability result already proved for the magnetic potentials. Our proof involves a Hodge decomposition as in Caro and Pohjola [8], see Lemma 6.2 therein. Our starting point is the following lemma. This is analogous to Lemma 5.1 in [8]. For this reason, we will only give the main ideas of the proof.

Lemma 3.7.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $B \subset \mathbb{R}^n$ be an open ball with $\overline{\Omega} \subset B$ and $\Gamma_0 \subset B \cap \{x \in \mathbb{R}^n : x_n = 0\}$. Let $A_1, A_2 \in L^\infty(\Omega; \mathbb{C}^n)$ and $q_1, q_2 \in L^\infty(\Omega; \mathbb{C})$. Given $M > 0$ and $s \in (0, 1/2)$, assume that $\chi_\Omega A_1$ and $\chi_\Omega A_2$ belong to $\mathcal{A}(\Omega, M, s)$. Let $\varphi \in W^{1,n}(B) \cap L^\infty(B)$ with $\varphi|_{\partial B} = 0$. Then for any $U_1, U_2 \in H^1(B)$ satisfying in B the equations:*

$$\mathcal{L}_{\chi_{\Omega\cup\Omega^*}\widetilde{A_1},\chi_{\Omega\cup\Omega^*}\widetilde{q_1}} U_1 = 0, \quad U_1|_{B\cap\{x\in\mathbb{R}^n:x_n=0\}} = 0, \tag{3.7.1}$$

$$\mathcal{L}_{\overline{\chi_{\Omega\cup\Omega^*}\widetilde{A_2}},\overline{\chi_{\Omega\cup\Omega^*}\widetilde{q_2}}} U_2 = 0, \quad U_2|_{B\cap\{x\in\mathbb{R}^n:x_n=0\}} = 0, \tag{3.7.2}$$

there exists a positive constant C (depending on $\Omega, n, M, s, \|q_1\|_{L^\infty}, \|q_2\|_{L^\infty}$) such that

$$\begin{aligned}
&\left|\int_B e^{i\varphi}\Big\{(\chi_{\Omega\cup\Omega^*}\widetilde{A_1} - (\chi_{\Omega\cup\Omega^*}\widetilde{A_2} + \nabla\varphi))\cdot\left(DU_1\overline{U_2} + U_1\overline{DU_2}\right)\right.\\
&\qquad + \Big[\chi_{\Omega\cup\Omega^*}\widetilde{A_1}^2 - (\chi_{\Omega\cup\Omega^*}\widetilde{A_2} + \nabla\varphi)^2 + \chi_{\Omega\cup\Omega^*}(\widetilde{q_1} - \widetilde{q_2})\\
&\qquad\quad \left. -(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2}) - \nabla\varphi)\cdot\nabla\varphi\Big] U_1\overline{U}_2\Big\}\right|\\
&\qquad \leqslant C\, dist\,(C_1^\Gamma, C_2^\Gamma)\, \|U_1\|_{H^1(\Omega\cup\Omega^*)}\, \|U_2\|_{H^1(\Omega\cup\Omega^*)}\,,
\end{aligned} \tag{3.7.3}$$

where C_j^Γ denotes the local Cauchy data set C_{A_j,q_j}^Γ for $j = 1, 2$.

Proof. Our proof starts by restricting (3.7.1)-(3.7.2) to Ω and Ω^* to obtain

$$\begin{aligned}
&\mathcal{L}_{\widetilde{A_1},\widetilde{q_1}}(U_1|_\Omega) = 0 \quad \text{in}\,\Omega, \quad (U_1|_\Omega)|_{\Gamma_0} = 0,\\
&\mathcal{L}_{\overline{\widetilde{A_2}},\overline{\widetilde{q_2}}}(U_2|_\Omega) = 0 \quad \text{in}\,\Omega, \quad (U_2|_\Omega)|_{\Gamma_0} = 0.
\end{aligned} \tag{3.7.4}$$

and

$$\begin{aligned}
&\mathcal{L}_{\widetilde{A_1},\widetilde{q_1}}(U_1|_{\Omega^*}) = 0 \quad \text{in}\,\Omega^*, \quad (U_1|_{\Omega^*})|_{\Gamma_0} = 0,\\
&\mathcal{L}_{\overline{\widetilde{A_2}},\overline{\widetilde{q_2}}}(U_2|_{\Omega^*}) = 0 \quad \text{in}\,\Omega^*, \quad (U_2|_{\Omega^*})|_{\Gamma_0} = 0.
\end{aligned} \tag{3.7.5}$$

Hence, by Corollary 3.2.2 applied to Ω and for the magnetic potentials $\widetilde{A_1}, \widetilde{A_2}$, the electric potentials $\widetilde{q_1}, \widetilde{q_2}$ and $U_1|_\Omega, U_2|_\Omega \in H^1(\Omega)$, we get

$$\begin{aligned}
&\left|\int_\Omega (\widetilde{A_1} - \widetilde{A_2})\cdot\left(DU_1\overline{U_2} + U_1\overline{DU_2}\right) + \left(\widetilde{A_1}^2 - \widetilde{A_2}^2 + \widetilde{q_1} - \widetilde{q_2}\right)U_1\overline{U_2}\right|\\
&\qquad \leqslant C_1\, dist\,(C_1^\Gamma, C_2^\Gamma)\, \|U_1\|_{H^1(\Omega)}\, \|U_2\|_{H^1(\Omega)}\\
&\qquad \leqslant C_1\, dist\,(C_1^\Gamma, C_2^\Gamma)\, \|U_1\|_{H^1(\Omega\cup\Omega^*)}\, \|U_2\|_{H^1(\Omega\cup\Omega^*)}\,.
\end{aligned} \tag{3.7.6}$$

Applying again Corollary 3.2.2 now to Ω^* for the magnetic potentials $\widetilde{A_1}, \widetilde{A_2}$, the electric potentials $\widetilde{q_1}, \widetilde{q_2}$ and $U_1|_{\Omega^*}, U_2|_{\Omega^*} \in H^1(\Omega^*)$, we get

$$\begin{aligned}
&\left|\int_{\Omega^*} (\widetilde{A_1} - \widetilde{A_2})\cdot\left(DU_1\overline{U_2} + U_1\overline{DU_2}\right) + \left(\widetilde{A_1}^2 - \widetilde{A_2}^2 + \widetilde{q_1} - \widetilde{q_2}\right)U_1\overline{U_2}\right|\\
&\qquad \leqslant C_2\, dist^*\,(C_1^\Gamma, C_2^\Gamma)\, \|U_1\|_{H^1(\Omega^*)}\, \|U_2\|_{H^1(\Omega^*)}\\
&\qquad \leqslant C_2\, dist^*\,(C_1^\Gamma, C_2^\Gamma)\, \|U_1\|_{H^1(\Omega\cup\Omega^*)}\, \|U_2\|_{H^1(\Omega\cup\Omega^*)}\,,
\end{aligned} \tag{3.7.7}$$

where $dist^*\,(C_1^\Gamma, C_2^\Gamma)$ is defined by $dist\,(C_1^\Gamma, C_2^\Gamma)$, see (3.1.5), with Ω replaced by Ω^*. Then, adding (3.7.6) and (3.7.7), we obtain

$$\begin{aligned}
&\left|\int_B \left[\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2})\right] \cdot \left(DU_1\overline{U_2} + U_1\overline{DU_2}\right)\right. \\
&\qquad\left. +\chi_{\Omega\cup\Omega^*}\left[\widetilde{A_1}^2 - \widetilde{A_2}^2 + \widetilde{q_1} - \widetilde{q_2}\right] U_1\overline{U_2}\right| \\
&\leqslant C_3\,(dist\,(C_1^\Gamma, C_2^\Gamma) + dist^*\,(C_1^\Gamma, C_2^\Gamma))\,\|U_1\|_{H^1(\Omega\cup\Omega^*)}\,\|U_2\|_{H^1(\Omega\cup\Omega^*)} \\
&= 2C_3\,dist\,(C_1^\Gamma, C_2^\Gamma)\,\|U_1\|_{H^1(\Omega\cup\Omega^*)}\,\|U_2\|_{H^1(\Omega\cup\Omega^*)}\,,
\end{aligned} \tag{3.7.8}$$

where we have used that $dist^*\,(C_1^\Gamma, C_2^\Gamma) = dist\,(C_1^\Gamma, C_2^\Gamma)$. On the other hand, by the gauge invariance of the Cauchy data sets, see Lemma 3.1 in [28], we get

$$\begin{aligned}
&\left|\int_B \left[\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2})\right] \cdot \left(DU_1\overline{U_2} + U_1\overline{DU_2}\right)\right. \\
&\qquad\left. +\chi_{\Omega\cup\Omega^*}\left[\widetilde{A_1}^2 - \widetilde{A_2}^2 + \widetilde{q_1} - \widetilde{q_2}\right] U_1\overline{U_2}\right| \\
&= \left\langle N_{\chi_{\Omega\cup\Omega^*}\widetilde{A_1},\chi_{\Omega\cup\Omega^*}\widetilde{q_1}} U_1, T_r U_2\right\rangle_B - \overline{\left\langle N_{\overline{\chi_{\Omega\cup\Omega^*}\widetilde{A_2}},\overline{\chi_{\Omega\cup\Omega^*}\widetilde{q_2}}} U_2, T_r U_1\right\rangle_B} \\
&= \left\langle N_{\chi_{\Omega\cup\Omega}\widetilde{A_1},\chi_{\Omega\cup\Omega^*}\widetilde{q_1}} U_1, T_r U_2\right\rangle_B \\
&\quad - \overline{\left\langle N_{\overline{\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\varphi},\overline{\chi_{\Omega\cup\Omega^*}\widetilde{q_2}}} (e^{-i\overline{\varphi}}U_2), T_r(e^{-i\varphi}U_1)\right\rangle_B} \\
&= \int_B e^{i\varphi}\left\{(\chi_{\Omega\cup\Omega^*}\widetilde{A_1} - (\chi_{\Omega\cup\Omega^*}\widetilde{A_2} + \nabla\varphi)) \cdot \left(DU_1\overline{U_2} + U_1\overline{DU}_2\right)\right. \\
&\quad + \left[\chi_{\Omega\cup\Omega^*}\widetilde{A_1}^2 - (\chi_{\Omega\cup\Omega^*}\widetilde{A_2} + \nabla\varphi)^2 + \chi_{\Omega\cup\Omega^*}(\widetilde{q_1} - \widetilde{q_2})\right. \\
&\qquad\left.\left. -(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2}) - \nabla\varphi) \cdot \nabla\varphi\right] U_1\overline{U}_2\right\},
\end{aligned} \tag{3.7.9}$$

where we have used that if $U_1 \in H^1(B)$ satisfies $\mathcal{L}_{\chi_{\Omega\cup\Omega^*}\widetilde{A_1},\chi_{\Omega\cup\Omega^*}\widetilde{q_1}} U_1 = 0$ in B then $e^{-i\varphi}U_1 \in H^1(B)$ satisfies $\mathcal{L}_{\chi_{\Omega\cup\Omega^*}\widetilde{A_1}+\nabla\varphi,\,\chi_{\Omega\cup\Omega^*}\widetilde{q_1}}(e^{-i\varphi}U_1) = 0$ in B. Analogously, if $U_2 \in H^1(B)$ satisfies $\mathcal{L}_{\overline{\chi_\Omega A_2},\overline{\chi_\Omega q_2}} U_2 = 0$ in B then $e^{-i\overline{\varphi}}U_2 \in H^1(B)$ satisfies $\mathcal{L}_{\overline{\chi_{\Omega\cup\Omega}\widetilde{A_2}+\nabla\varphi},\,\overline{\chi_{\Omega\cup\Omega}\widetilde{q_2}}}(e^{-i\overline{\varphi}}U_2) = 0$ in B. Thus, from Lemma 3.2.1 with Ω replaced by B and replacing (3.7.9) into (3.7.8), we end the proof. $\square$

To estimate stability estimates for the magnetic potentials, we used Corollary 3.2.2 in order to isolate $A_1 - A_2$, and then using solutions for the Schrödinger operator $\mathcal{L}_{A,q}u = 0$, given by Proposition 3.4.3 we obtained the estimate (3.5.1) of Proposition 3.5.1. We follow similar ideas. Our task will be to isolate $\chi_{\Omega\cup\Omega^*}(\widetilde{q_1} - \widetilde{q_2})$ from the left-hand side of (3.7.3) and taking into account two facts. The first fact will be to obtain an estimate for the function $\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2}) - \nabla\varphi$, where the function φ will be fixed later on. This can be done by using the following Hodge decomposition derived in [8]. To have an idea of the different sets declared in the following lemma, see Figure 3.2 below.

Lemma 3.7.2. *Let $B \subset \mathbb{R}^n$ be an open ball satisfying $\overline{\Omega} \subset B$. Let A_1 and A_2 belong to $L^\infty(\Omega;\mathbb{C}^n)$. Then there exist $\psi \in W^{1,p}(B)$ with $p \geqslant 2$ and $C > 0$ satisfying the following conditions:*

$$\|\psi\|_{W^{1,p}(B)} \leqslant C \left\|\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2})\right\|_{L^p(\mathbb{R}^n)} \tag{3.7.10}$$

and

$$\left\|\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2}) - \nabla\psi\right\|_{L^2(B)} \leqslant C \left\|d(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2}))\right\|_{H^{-1}(B)}. \tag{3.7.11}$$

Moreover, if B' is another open ball with $\overline{\Omega} \subset B' \subset\subset B$ then

$$\|\psi - \underline{\psi}\|_{H^1(B\setminus\overline{B'})} \leqslant C \left\|d(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2}))\right\|_{H^{-1}(B)}, \tag{3.7.12}$$

where $\underline{\psi}$ denotes the average of ψ in $B \setminus \overline{B'}$.

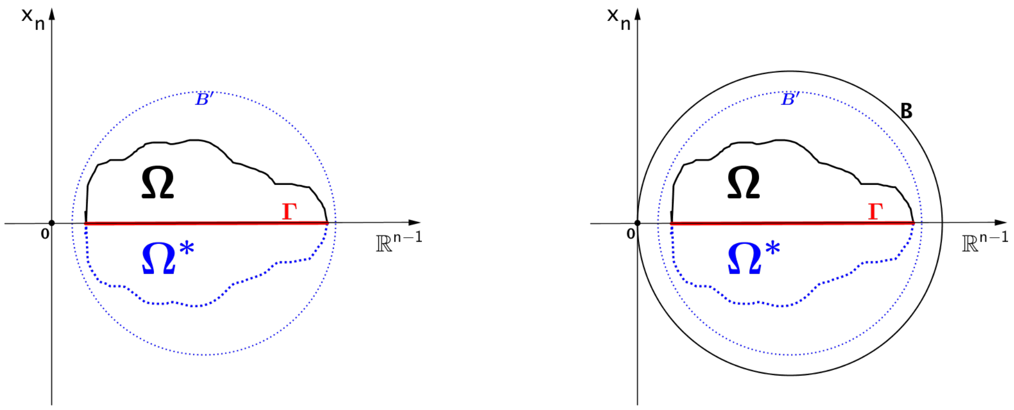

Figure 3.2: Shape of the sets Ω and Ω^*. Also the balls B' and B.

The second fact will be to use the solutions $U_j \in H^1(B)$ with $j = 1,2$ (with the requirements of Lemma 3.7.1) constructed implicitly in the proof of Proposition 3.4.3.

3.7.1 A Fourier estimate for the electric potentials

In this section we prove following proposition. This is analogous to Proposition 3.5.1.

Proposition 3.7.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $B' \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ be an open balls with $\overline{\Omega \cup \Omega^*} \subset B' \subset\subset B$. Let $A_1, A_2 \in L^\infty(\Omega, \mathbb{C}^n)$ and $q_1, q_2 \in L^\infty(\Omega, \mathbb{C})$. Given $M > 0$ and $s \in (0, 1/2)$, assume that $\chi_\Omega A_1, \chi_\Omega A_2 \in \mathcal{A}(\Omega, M, s)$ and $\chi_\Omega q_1, \chi_\Omega q_2 \in \mathcal{Q}(\Omega, M, s)$. Then there exist three positive constants C, τ_0 and ε_0 (all depending on*

Ω, n, M, s) such that the following estimate

$$
\begin{aligned}
&|\mathcal{F}\left[\chi_{\Omega\cup\Omega^*}\widetilde{q_1}\right](\xi) - \mathcal{F}\left[\chi_{\Omega\cup\Omega^*}\widetilde{q_2}\right](\xi)| \\
&\leqslant C\left(e^{2\tau\kappa}dist\,(C_1^\Gamma, C_2^\Gamma) + \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{\theta s^2}{(s+2)(n+ns+2s)}}\tau^{(s+4)/(s+2)}\right. \\
&\qquad\qquad \left. +\tau^{-s/(s+2)} + e^{-4\pi\varepsilon^2\tau^2\frac{|\xi'|^2}{|\xi|^2}} + \varepsilon^s\right),
\end{aligned} \tag{3.7.13}
$$

holds true for every $\xi \in \bigcap_{l=1}^{n-1} E_l$ (see (3.4.18)), for all $\tau \geqslant \tau_0$ and for all $0 < \varepsilon < \varepsilon_0$. Here $\theta \in (0, 2/n)$.

Proof. The proof is similar to the proof of Lemma 5.3 in [8]. Consider the function ψ given by Lemma 3.7.2 with $p > n$. Notice that this function does not necessarily satisfy the vanishing condition on ∂B. We will remedy this by using a cutoff argument. So consider a smooth function $\chi \in C_0^\infty(B)$ such that $\chi(x) = 1$ in B' and set $\varphi = \chi(\psi - \underline{\psi})$. Note that $\varphi|_{\partial B} = 0$. Thus, by (3.7.10), Morrey's inequality and the boundedness of B we get

$$
\|\varphi\|_{L^\infty(B)} + \|\nabla\varphi\|_{L^n(B)} + \|\nabla\psi\|_{L^n(B)} \leqslant C_1. \tag{3.7.14}
$$

Now we compute the left-hand side of (3.7.3) by using the functions $U_1, U_2 \in H^1(B)$ satisfying in B: $\mathcal{L}_{\chi_{\Omega\cup\Omega^*}\widetilde{A_1},\,\chi_{\Omega\cup\Omega^*}\widetilde{q_1}}U_1 = 0$ with $U_1|_{\partial B^+\cap\{x_n=0\}} = 0$ and $\mathcal{L}_{\overline{\chi_{\Omega\cup\Omega^*}\widetilde{A_2}},\overline{\chi_{\Omega\cup\Omega^*}\widetilde{q_2}}}U_2 = 0$ with $U_2|_{\partial B^+\cap\{x_n=0\}} = 0$. Such functions were constructed implicitly in the proof of Proposition 3.4.3. Also consider ρ_1 and ρ_2 defined by (3.4.12) with μ_1 and μ_2 declared in Lemma 3.4.4. Now we will divide the proof into three steps. The first step will be to prove the following claim.

Claim 1

If $\theta \in (0, 2/n)$ then there exist a constant $C_2 > 0$ such that

$$
\begin{aligned}
&\left|\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1} - \widetilde{q_2})U_1\overline{U_2}\right| \\
&\leqslant C_2\left(e^{2\tau\kappa}dist\,(C_1^\Gamma, C_2^\Gamma) + \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{\theta s^2}{(s+2)(n+ns+2s)}}\tau^{(s+4)/(s+2)}\right),
\end{aligned} \tag{3.7.15}
$$

for every $\xi \in \bigcap_{l=1}^{n-1} E_l$. In fact, adding and subtracting the same terms we have the obvious identity:

$$
\begin{aligned}
&\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})U_1\overline{U_2}\\
&=\int_B e^{i\varphi}(\chi_{\Omega\cup\Omega^*}\widetilde{A_1}-(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\varphi))\cdot\left(DU_1\overline{U_2}+U_1\overline{DU}_2\right)\\
&+\int_B e^{i\varphi}\Big(\chi_{\Omega\cup\Omega^*}A_1^2-(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\varphi)^2+\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})\\
&\qquad -(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2})-\nabla\varphi)\cdot\nabla\varphi\Big)U_1\overline{U}_2\\
&-\int_B e^{i\varphi}(\chi_{\Omega\cup\Omega^*}\widetilde{A_1}-(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\varphi))\cdot\left(DU_1\overline{U_2}+U_1\overline{DU}_2\right)\\
&-\int_B e^{i\varphi}\Big[\chi_{\Omega\cup\Omega^*}\widetilde{A_1}^2-(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\varphi)^2\\
&\qquad\qquad -(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2})-\nabla\varphi)\cdot\nabla\varphi\Big]U_1\overline{U}_2.
\end{aligned}
\tag{3.7.16}
$$

For convenience and as in [8], we set $\varphi' := (1-\chi)(\psi-\underline{\psi})$. Then we have $\psi-\underline{\psi} = \chi(\psi-\underline{\psi})+(1-\chi)(\psi-\underline{\psi}) = \varphi+\varphi'$, which in turn implies that

$$\nabla\psi = \nabla\varphi+\nabla\varphi'. \tag{3.7.17}$$

From this identity, it follows that

$$\chi_{\Omega\cup\Omega^*}\widetilde{A_1}-(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\psi) = \chi_{\Omega\cup\Omega^*}\widetilde{A_1}-(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\varphi)-\nabla\varphi' \tag{3.7.18}$$

and

$$
\begin{aligned}
&\chi_{\Omega\cup\Omega^*}\widetilde{A_1}^2-(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\psi)^2\\
&=\left[\chi_{\Omega\cup\Omega^*}\widetilde{A_1}-(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\psi)\right]\cdot\left[\chi_{\Omega\cup\Omega^*}\widetilde{A_1}+(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\psi)\right]\\
&=\left[\chi_{\Omega\cup\Omega^*}\widetilde{A_1}-(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\varphi)-\nabla\varphi'\right]\\
&\qquad\qquad\cdot\left[\chi_{\Omega\cup\Omega^*}\widetilde{A_1}+(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\varphi)+\nabla\varphi'\right]\\
&=\chi_{\Omega\cup\Omega^*}\widetilde{A_1}^2-(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\varphi)^2+\left[\chi_{\Omega\cup\Omega^*}\widetilde{A_1}-(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\varphi)\right]\cdot\nabla\varphi'\\
&\quad-\left[\chi_{\Omega\cup\Omega^*}\widetilde{A_1}+(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\varphi)\right]\cdot\nabla\varphi'-\nabla\varphi'\cdot\nabla\varphi'.
\end{aligned}
\tag{3.7.19}
$$

Hence, replacing (3.7.17)-(3.7.19) into (3.7.16) and by a straightforward computation, we obtain

$$\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})U_1\overline{U_2} := I+II+III+IV,$$

where

$$I := -\int_B e^{i\varphi}\left(\chi_{\Omega\cup\Omega^*}\widetilde{A_1}^2-(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\psi)^2\right)U_1\overline{U_2}, \tag{3.7.20}$$

$$II : \int_B e^{i\varphi} \left(\chi_{\Omega\cup\Omega^*}\widetilde{A_1} - \chi_{\Omega\cup\Omega^*}\widetilde{A_2} - \nabla\psi\right) \cdot \nabla\varphi \, U_1\overline{U_2}, \tag{3.7.21}$$

$$III := -\int_B e^{i\varphi} \left(\chi_{\Omega\cup\Omega^*}\widetilde{A_1} - \chi_{\Omega\cup\Omega^*}\widetilde{A_2} - \nabla\psi\right) \cdot (DU_1\overline{U_2} + U_1\overline{DU_2}), \tag{3.7.22}$$

and

$$\begin{aligned} IV := & \int_B e^{i\varphi}(\chi_{\Omega\cup\Omega^*}\widetilde{A_1} - \chi_{\Omega\cup\Omega^*}\widetilde{A_2} - \nabla\psi) \cdot (DU_1\overline{U_2} + U_1\overline{DU_2}) \\ & + \int_B e^{i\varphi} \Big[\chi_{\Omega\cup\Omega^*}\widetilde{A_1}^2 - (\chi_{\Omega\cup\Omega^*}\widetilde{A_2} + \nabla\psi)^2 + \chi_{\Omega\cup\Omega^*}(\widetilde{q_1} - \widetilde{q_2}) \\ & \qquad -(\chi_{\Omega\cup\Omega^*}\widetilde{A_1} - \chi_{\Omega\cup\Omega^*}\widetilde{A_2} - \nabla\psi) \cdot \nabla\varphi\Big] \, U_1\overline{U_2}. \end{aligned} \tag{3.7.23}$$

Then, by the triangle inequality, it is immediate that

$$\left|\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1} - \widetilde{q_2})U_1\overline{U_2}\right| \leqslant |I| + |II| + |III| + |IV| \, . \tag{3.7.24}$$

Now we will estimate the terms from $|I|$ to $|IV|$. From (3.5.3)-(3.5.4) with $\Omega\cup\Omega^*$ replaced by B and taking into account (3.4.25), we have

$$\begin{aligned} U_1\overline{U_2} = & \, e^{i\xi\cdot x}(e^{\Phi_1^\sharp + \overline{\Phi_2^\sharp}} + e^{\Phi_1^\sharp}\overline{r_2} + e^{\overline{\Phi_2^\sharp}}r_1 + r_1\overline{r_2}) \\ & - e^{i\left(\xi', 2\sqrt{\tau^2 - \frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)}(e^{\Phi_1^\sharp + \overline{\Phi_2^{\sharp\,*}}} + e^{\Phi_1^\sharp}\overline{r_2^*} + e^{\overline{\Phi_2^{\sharp\,*}}} r_1 + r_1\overline{r_2^*}) \\ & - e^{i\left(\xi', -2\sqrt{\tau^2 - \frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)}(e^{\Phi_1^{\sharp\,*} + \overline{\Phi_2^{\sharp\,*}}} + e^{\Phi_1^{\sharp\,*}}\overline{r_2} + e^{\overline{\Phi_2^\sharp}}r_1^* + r_1^*\overline{r_2}) \\ & + e^{i\xi^*\cdot x}(e^{\Phi_1^{\sharp\,*} + \overline{\Phi_2^{\sharp\,*}}} + e^{\Phi_1^{\sharp\,*}}\overline{r_2^*} + e^{\overline{\Phi_2^{\sharp\,*}}} r_1^* + r_1^*\overline{r_2^*}). \end{aligned} \tag{3.7.25}$$

For $j = 1, 2$, from (3.5.6) with $\Omega\cup\Omega^*$ replaced by B and a standard Sobolev's embedding, we deduce that there exist two positive constant C_2 and C_3 such that

$$\begin{aligned} & \|r_j\|_{L^{2n/(n-2)}(B)} + \left\|r_j^*\right\|_{L^{2n/(n-2)}(B)} \\ & \leqslant C_2 \left(\|r_j\|_{W^{1,2}(B)} + \left\|r_j^*\right\|_{W^{1,2}(B)}\right) \\ & \leqslant C_3 \, \tau^{2/(s+2)}. \end{aligned} \tag{3.7.26}$$

Also, from (3.5.5) and the boundedness of B, we get

$$\left\|e^{\Phi_j^\sharp}\right\|_{L^{2n/(n-2)}(B)} + \left\|e^{\Phi_j^{\sharp\,*}}\right\|_{L^{2n/(n-2)}(B)} \leqslant C_4. \tag{3.7.27}$$

Thus, from (3.7.14) and the boundedness of B it follows that $\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2}) - \nabla\psi \in L^n(B)$. For similar reasons, we have that $\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2}) + \nabla\psi \in L^n(B)$. Hence, from

(3.7.14), (3.7.26)-(3.7.27), the boundedness of B and applying Hölder's inequality for $1/n+1/n+(n-2)/(2n)+(n-2)/(2n)=1$, we get

$$\begin{aligned}|I| &= \bigg|\int_B e^{i\varphi}(\chi_{\Omega\cup\Omega^*}\widetilde{A_1}-(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\psi)) \\ &\qquad\qquad \cdot(\chi_{\Omega\cup\Omega^*}\widetilde{A_1}+(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\psi))U_1\overline{U_2}\bigg|. \\ &\leqslant C_5\left\|\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2})-\nabla\psi\right\|_{L^n(B)}\tau^{4/(s+2)}.\end{aligned} \tag{3.7.28}$$

To estimate the second term we use that $\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2})-\nabla\psi\in L^n(B)$. Moreover, from (3.7.14) we have that $\nabla\varphi\in L^n(B)$. Thus, applying Hölder's inequality for $1/n+1/n+(n-2)/(2n)+(n-2)/(2n)=1$, we get

$$|II|\leqslant C_6\left\|\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2})-\nabla\psi\right\|_{L^n(B)}\tau^{4/(s+2)}. \tag{3.7.29}$$

To estimate the terms III and IV we take into account the computations done in (3.5.12). Thus, for $j=1,2$, from (3.7.26), (3.5.6) with $\Omega\cup\Omega^*$ replaced by B and Hölder's inequality applied to $1/n+(n-2)/(2n)+1/2=1$, we obtain

$$|III|\leqslant C_7\left\|\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2})-\nabla\psi\right\|_{L^n(B)}\tau^{(s+4)/(s+2)}. \tag{3.7.30}$$

The estimate for IV requires a more delicate analysis. Replacing (3.7.18)-(3.7.19) into (3.7.23) and by a straightforward computation, we get the following identity:

$$\begin{aligned}IV &:= \int_B e^{i\varphi}(\chi_{\Omega\cup\Omega^*}\widetilde{A_1}-\chi_{\Omega\cup\Omega^*}\widetilde{A_2}-\nabla\varphi)\cdot(DU_1\overline{U_2}+U_1\overline{DU_2}) \\ &\quad+\int_B e^{i\varphi}\Big[\chi_{\Omega\cup\Omega^*}\widetilde{A_1}^2-(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}+\nabla\varphi)^2+\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2}) \\ &\qquad\qquad -(\chi_{\Omega\cup\Omega^*}\widetilde{A_1}-\chi_{\Omega\cup\Omega^*}\widetilde{A_2}-\nabla\varphi)\cdot\nabla\varphi\Big]U_1\overline{U_2} \\ &\quad-\int_B e^{i\varphi}\nabla\varphi'\cdot(DU_1\overline{U_2}+U_1\overline{DU_2}) \\ &\quad-\int_B e^{i\varphi}\left(2\chi_{\Omega\cup\Omega^*}\widetilde{A_2}\cdot\nabla\varphi'+\nabla\psi\cdot\nabla\varphi'\right)U_1\overline{U_2}.\end{aligned}$$

Notice that since the functions $\chi_{\Omega\cup\Omega^*}\widetilde{A_2}$ and $\nabla\varphi'$ have disjoint supports, it follows that $\int_B e^{i\varphi}(\chi_{\Omega\cup\Omega^*}\widetilde{A_2})\cdot\nabla\varphi'=0$. Hence, Lemma 3.7.1 and the triangular inequality imply that

$$\begin{aligned}|IV| &\leqslant C_8\Big(dist\,(C_1^\Gamma,C_2^\Gamma)\,\|U_1\|_{H^1(\Omega\cup\Omega^*)}\,\|U_2\|_{H^1(\Omega\cup\Omega^*)} \\ &\qquad+\left|\int_B e^{i\varphi}\nabla\varphi'\cdot(U_1\overline{DU_2}+DU_1\overline{U_2})+\nabla\psi\cdot\nabla\varphi' U_1\overline{U_2}\right|\Big).\end{aligned}$$

Once again applying Hölder's inequality and by similar arguments as used to estimate $U_1\overline{DU_2} + DU_1\overline{U_2}$ and $U_1\overline{U_2}$, we get

$$
\begin{aligned}
|IV| \leqslant C_9 \Big(& dist\,(C_1^\Gamma, C_2^\Gamma)\, \|U_1\|_{H^1(\Omega\cup\Omega^*)}\, \|U_2\|_{H^1(\Omega\cup\Omega^*)} \\
&+ \left\|\nabla\varphi'\right\|_{L^n(B)}\, \tau^{(s+4)/(s+2)}\Big).
\end{aligned}
\tag{3.7.31}
$$

Furthermore, an elementary interpolation, the boundedness of B, (3.7.11) and the estimate (3.6.10), we have

$$
\begin{aligned}
&\left\|\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2}) - \nabla\psi\right\|_{L^n(B)} \\
&\leqslant \left\|\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2}) - \nabla\psi\right\|_{L^2(B)}^{\theta} \left\|\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2}) - \nabla\psi\right\|_{L^p(B)}^{1-\theta} \\
&\qquad\leqslant C_{10} \left\|d(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2}))\right\|_{H^{-1}(B)}^{\theta} \\
&\qquad\leqslant C_{10} \left\|d(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2}))\right\|_{H^{-1}(\mathbb{R}^n)}^{\theta} \\
&\qquad\leqslant C_{11} \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{\theta s^2}{(s+2)(n+ns+2s)}},
\end{aligned}
\tag{3.7.32}
$$

for every $\theta \in (0, 2/n)$ and p is chosen such that $\theta/2 + (1-\theta)/p = 1/n$. Notice that since $\varphi' = (1-\chi)(\psi - \psi^*)$, we deduce that

$$
\left\|\nabla\varphi'\right\|_{L^n(B)} \leqslant C_7 \|\psi - \psi^*\|_{W^{1,n}(B\setminus\overline{B'})}.
$$

Again by an elementary interpolation, (3.7.12), the boundedness of B, Theorem 1.2.4 and since $1-\chi \equiv 0$ in B', we get

$$
\begin{aligned}
\left\|\nabla\varphi'\right\|_{L^n(B)} &\leqslant C_{12} \|\psi - \psi^*\|_{H^1(B\setminus\overline{B'})}^{\theta} \|\psi - \psi^*\|_{W^{1,p}(B\setminus\overline{B'})}^{1-\theta} \\
&\leqslant C_{13} \left\|d(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2}))\right\|_{H^{-1}(B)}^{\theta} \\
&\leqslant C_{13} \left\|d(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2}))\right\|_{H^{-1}(\mathbb{R}^n)}^{\theta} \\
&\leqslant C_{14} \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{\theta s^2}{(s+2)(n+ns+2s)}}.
\end{aligned}
\tag{3.7.33}
$$

Hence, using estimate (3.7.32) into (3.7.28)-(3.7.30) we get

$$
|I| + |II| + |III| \leqslant C_{15} \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{\theta s^2}{(s+2)(n+ns+2s)}} \tau^{(s+4)/(s+2)}.
$$

Using estimate (3.7.33) into (3.7.31) we obtain

$$
|IV| \leqslant C_{16}(e^{2\tau k} dist\,(C_1^\Gamma, C_2^\Gamma) + \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{\theta s^2}{(s+2)(n+ns+2s)}} \tau^{(s+4)/(s+2)}).
$$

We conclude the proof of Claim 1 by combining the two above estimates into (3.7.24). The second step will be to prove the following claim.

Claim 2

There exists $C_{17} > 0$ such that

$$\begin{aligned}&\left|\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})e^{i\xi\cdot x}e^{\Phi_1^\sharp+\overline{\Phi_2^\sharp}}+\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})e^{i\xi^*\cdot x}e^{{\Phi_1^\sharp}^*+\overline{{\Phi_2^\sharp}^*}}\right|\\ &\leqslant C_{17}\left(\left|\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})U_1\overline{U_2}\right|\right.\\ &\qquad\left.+\tau^{-s/(s+2)}+\left|\log dist\,(C_1^\Gamma,C_2^\Gamma)\right|^{-\frac{s^2}{(s+2)(n+ns+2s)}}+e^{-4\pi\varepsilon^2\tau^2\frac{|\xi'|^2}{|\xi|^2}}+\varepsilon^s\right),\end{aligned}\tag{3.7.34}$$

for every $\xi\in\bigcap_{l=1}^{n-1}E_l$. In fact, the idea will be to isolate the function $\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})$ from the left-hand side of (3.7.15). For convenience, we set

$$V=\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})e^{i\xi^*\cdot x}\left(e^{{\Phi_1^\sharp}^*}\overline{r_2^*}+e^{\overline{{\Phi_2^\sharp}^*}}r_1^*+r_1^*\overline{r_2^*}\right),$$

$$VI=\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})e^{i\xi\cdot x}\left(e^{\Phi_1^\sharp}\overline{r_2}+e^{\overline{\Phi_2^\sharp}}r_1+r_1\overline{r_2}\right),$$

$$\begin{aligned}VII=\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})&e^{i\left(\xi',2\tau\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)}\\ &\times\left(e^{\Phi_1^\sharp}\overline{r_2^*}+e^{\overline{{\Phi_2^\sharp}^*}}r_1+r_1\overline{r_2^*}\right),\end{aligned}$$

$$\begin{aligned}VIII=\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})&e^{i\left(\xi',-2\tau\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)}\\ &\times\left(e^{{\Phi_1^\sharp}^*}\overline{r_2}+e^{\overline{{\Phi_2^\sharp}^*}}r_1^*+r_1^*\overline{r_2}\right),\end{aligned}$$

$$IX=\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})e^{i\left(\xi',2\tau\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)}e^{\Phi_1^\sharp+\overline{\Phi_2^\sharp}},$$

$$X=\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})e^{i\left(\xi',-2\tau\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)}e^{{\Phi_1^\sharp}^*+\overline{{\Phi_2^\sharp}^*}}.$$

Thus, adding and subtracting terms we obtain

$$\begin{aligned}&\left|\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})e^{i\xi\cdot x}e^{\Phi_1^\sharp+\overline{\Phi_2^\sharp}}+\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})e^{i\xi^*\cdot x}e^{{\Phi_1^\sharp}^*+\overline{{\Phi_2^\sharp}^*}}\right|\\ &\leqslant\left|\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})U_1\overline{U_2}\right|+|V+VI+VII+VIII+IX+X|.\end{aligned}\tag{3.7.35}$$

Now the task is to estimate each term of the right-hand side of the above inequality. From (3.5.6) with $\Omega\cup\Omega^*$ replaced by B, (3.7.14), the boundedness of B and Hölder's inequality in $L^2(B)$, we get

$$|V+VI+VII+VIII| \leqslant C_{18}\,\tau^{-s/(s+2)}. \tag{3.7.36}$$

The estimate for IX and X require a more delicate analysis. Adding and subtracting terms it is easy to check that

$$\begin{aligned}|IX| \leqslant & \left|\int_B \chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})e^{i\left(\xi',2\tau\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)}\right| \\ & + \left|\int_B \chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})e^{i\left(\xi',2\tau\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)}(1-e^{\Phi_1+\overline{\Phi_2}+i\varphi})\right| \\ & + \left|\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})e^{i\left(\xi',2\tau\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)}(e^{\Phi_1+\overline{\Phi_2}}-e^{\Phi_1^\sharp+\overline{\Phi_2^\sharp}})\right|.\end{aligned} \tag{3.7.37}$$

On one hand, by Lemma 3.4.1 we have $\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2}) \in B_s^{2,\infty}(\mathbb{R}^n)$. Thus, by Lemma 3.5.3 applied to $f=\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})$, $C_0=\|\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})\|_{B_s^{2,\infty}}$ and $\sigma=1$, we obtain

$$\left|\int_B \chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})e^{i\left(\xi',2\tau\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)}\right| \leqslant C_{19}\left(e^{-4\pi\varepsilon^2\tau^2\frac{|\xi'|^2}{|\xi|^2}}+\varepsilon^s\right). \tag{3.7.38}$$

On the other hand, from (3.5.32) we deduce that

$$(i\mu_1+\mu_2)\cdot\nabla(\Phi_1+\overline{\Phi_2}+i\varphi) = (\mu_1-i\mu_2)\cdot(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2})-\nabla\varphi). \tag{3.7.39}$$

Thus, by the boundedness of $((i\mu_1+\mu_2)\cdot\nabla)^{-1}$ in weighted L^2 spaces, the estimates (3.7.11) and (3.7.12), we obtain

$$\begin{aligned}\left\|\Phi_1+\overline{\Phi_2}+i\varphi\right\|_{L^2(B)} &\leqslant C_{19}\left\|\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2})-\nabla\varphi\right\|_{L^2(B)} \\ &\leqslant C_{20}\left(\left\|\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2})-\nabla\psi\right\|_{L^2(B)}+\|\psi-\psi^*\|_{H^1(B\setminus\overline{B'})}\right) \\ &\leqslant C_{21}\left|\log dist\,(C_1^\Gamma,C_2^\Gamma)\right|^{-\frac{s^2}{(s+2)(n+ns+2s)}}.\end{aligned} \tag{3.7.40}$$

From (3.5.21), the boundedness of B and the above inequality, we have

$$\begin{aligned}&\left|\int_B \chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})e^{i\left(\xi',2\tau\sqrt{1-\tau^{-2}\frac{|\xi|^2}{4}}\frac{|\xi'|}{|\xi|}\right)}\left(1-e^{\Phi_1+\overline{\Phi_2}+i\varphi}\right)\right| \\ &\leqslant C_{22}\,\|\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})\|_{L^2(B)}\left\|\Phi_1+\overline{\Phi_2}+i\varphi\right\|_{L^2(B)} \\ &\leqslant C_{23}\left|\log dist\,(C_1^\Gamma,C_2^\Gamma)\right|^{-\frac{s^2}{(s+2)(n+ns+2s)}}.\end{aligned} \tag{3.7.41}$$

From (3.5.11), (3.5.21), (3.7.14) and following similar arguments to estimate (3.5.22), we get

$$
\begin{aligned}
&\left|\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})e^{i(\xi',2\tau|\xi'|)}(e^{\Phi_1+\overline{\Phi_2}}-e^{\Phi_1^\sharp+\overline{\Phi_2^\sharp}})\right| \\
&\leqslant \left\|e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})\right\|_{L^2(B)}\left\|\Phi_1-\Phi_1^\sharp+\overline{\Phi_2}-\Phi_2^\sharp\right\|_{L^2(B)} \\
&\leqslant C_{24}\,\tau^{-s/(s+2)}.
\end{aligned}
\tag{3.7.42}
$$

Hence, by replacing (3.7.38)-(3.7.42) into (3.7.37), we obtain

$$
|IX|\leqslant C_{25}(\tau^{-s/(s+2)}+\left|\log dist\,(C_1^\Gamma,C_2^\Gamma)\right|^{-\frac{s^2}{(s+2)(n+ns+2s)}}+e^{-4\pi\varepsilon^2\tau^2\frac{|\xi'|^2}{|\xi|^2}}+\varepsilon^s).
\tag{3.7.43}
$$

Now we estimate the term X. From (3.5.32) and (3.7.39) we deduce

$$
(i\mu_1^*+\mu_2^*)\cdot\nabla(\Phi_1^*+\overline{\Phi_2^*}+i\varphi)=(\mu_1^*-i\mu_2^*)\cdot(\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2})-\nabla\varphi),
\tag{3.7.44}
$$

which implies that

$$
\begin{aligned}
&\left\|\Phi_1^*+\overline{\Phi_2^*}+i\varphi\right\|_{L^2(B)}\leqslant C_{19}\left\|\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2})-\nabla\varphi\right\|_{L^2(B)} \\
&\leqslant C_{20}\left(\left\|\chi_{\Omega\cup\Omega^*}(\widetilde{A_1}-\widetilde{A_2})-\nabla\psi\right\|_{L^2(B)}+\|\psi-\psi^*\|_{H^1(B\backslash\overline{B'})}\right) \\
&\leqslant C_{21}\left|\log dist\,(C_1^\Gamma,C_2^\Gamma)\right|^{-\frac{s^2}{(s+2)(n+ns+2s)}}.
\end{aligned}
\tag{3.7.45}
$$

Thus, by similar reasoning applied to estimate IX, we can deduce that

$$
|X|\leqslant C_{26}(\tau^{-s/(s+2)}+\left|\log dist\,(C_1^\Gamma,C_2^\Gamma)\right|^{-\frac{s^2}{(s+2)(n+ns+2s)}}+e^{-4\pi\varepsilon^2\tau^2\frac{|\xi'|^2}{|\xi|^2}}+\varepsilon^s).
\tag{3.7.46}
$$

We conclude the proof of Claim 2 by combining the estimates (3.7.36) and (3.7.43)-(3.7.46) into (3.7.35).

The last step will be to use the estimates from the claims 1 and 2 in order to deduce the assertion of this proposition. By a straightforward computation, adding and subtracting terms, for every $\xi\in\bigcap_{l=1}^{n-1}E_l$, we can deduce that

$$
\begin{aligned}
&\left|\mathcal{F}\left[\chi_{\Omega\cup\Omega^*}\widetilde{q_1}\right](\xi)-\mathcal{F}\left[\chi_{\Omega\cup\Omega^*}\widetilde{q_2}\right](\xi)\right| \\
&=\left|\int_B\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})e^{i\xi\cdot x}+\int_B\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})e^{i\xi^*\cdot x}\right| \\
&:=\left|M_1+M_2+M_3+M_4+M_5\right|,
\end{aligned}
\tag{3.7.47}
$$

where

$$
M_1=\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})e^{i\xi\cdot x}e^{\Phi_1^\sharp+\overline{\Phi_2^\sharp}}+\int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1}-\widetilde{q_2})e^{i\xi^*\cdot x}e^{{\Phi_1^\sharp}^*+\overline{{\Phi_2^\sharp}^*}},
$$

$$M_2 = \int_B \chi_{\Omega\cup\Omega^*}(\widetilde{q_1} - \widetilde{q_2})e^{i\xi\cdot x}(1 - e^{\Phi_1+\overline{\Phi_2}+i\varphi}),$$
$$M_3 = \int_B \chi_{\Omega\cup\Omega^*}(\widetilde{q_1} - \widetilde{q_2})e^{i\xi^*\cdot x}(1 - e^{\Phi_1^*+\overline{\Phi_2^*}+i\varphi}),$$
$$M_4 = \int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1} - \widetilde{q_2})e^{i\xi\cdot x}(e^{\Phi_1+\overline{\Phi_2}} - e^{\Phi_1^\sharp+\overline{\Phi_2^\sharp}})$$

and

$$M_5 = \int_B e^{i\varphi}\chi_{\Omega\cup\Omega^*}(\widetilde{q_1} - \widetilde{q_2})e^{i\xi^*\cdot x}(e^{\Phi_1^*+\overline{\Phi_2^*}} - e^{\Phi_1^{\sharp *}+\overline{\Phi_2^{\sharp *}}}).$$

The task is now to estimate each one of the above expressions. From the claims 1 and 2, we obtain

$$\begin{aligned}|M_1| \leqslant C_{26}(e^{2\tau\kappa} dist\,(C_1^\Gamma, C_2^\Gamma) + \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{\theta s^2}{(s+2)(n+ns+2s)}}\,\tau^{(s+4)/(s+2)} \\ + \tau^{-s/(s+2)} + \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{s^2}{(s+2)(n+ns+2s)}} + e^{-4\pi\varepsilon^2\tau^2\frac{|\xi'|^2}{|\xi|^2}} + \varepsilon^s).\end{aligned} \tag{3.7.48}$$

From the boundedness of B, (3.5.21) and (3.7.40), we get

$$\begin{aligned}|M_2| &\leqslant C_{27}\left\|\chi_{\Omega\cup\Omega^*}(\widetilde{q_1} - \widetilde{q_2})\right\|_{L^2(B)}\left\|\Phi_1 + \overline{\Phi_2} + i\varphi\right\|_{L^2(B)} \\ &\leqslant C_{28}\left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{s^2}{(s+2)(n+ns+2s)}}.\end{aligned} \tag{3.7.49}$$

By similar arguments and from (3.7.45), we have

$$|M_3| \leqslant C_{29}\left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{s^2}{(s+2)(n+ns+2s)}}. \tag{3.7.50}$$

From (3.5.11), (3.5.21), (3.7.14) and as were estimated (3.5.22) and (3.7.42), we get

$$|M_4 + M_5| \leqslant C_{30}\,\tau^{-s/(s+2)}. \tag{3.7.51}$$

We conclude the proof of this proposition by replacing (3.7.48)-(3.7.51) into (3.7.47). □

3.7.2 Proof of Theorem 1.2.5

The proof of Theorem 1.2.5 is standard and similar to the proof of Theorem 1.2.4. By Proposition 3.5.1, taking into account that the constant $C > 0$ in the estimate (3.5.1) is independent of $\xi \in \bigcap_{l=1}^{n-1} E_l$ and since the set $\bigcap_{l=1}^{n-1} E_l$ is dense in $\mathbb{R}^n$, it follows that the following estimate

$$\begin{aligned}&\left|\mathcal{F}\left[\chi_{\Omega\cup\Omega^*}\widetilde{q_1}\right](\xi) - \mathcal{F}\left[\chi_{\Omega\cup\Omega^*}\widetilde{q_2}\right](\xi)\right| \\ &\leqslant C\left(e^{2\tau\kappa} dist\,(C_1^\Gamma, C_2^\Gamma) + \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{\theta s^2}{(s+2)(n+ns+2s)}}\,\tau^{(s+4)/(s+2)}\right. \\ &\qquad\left. + \tau^{-s/(s+2)} + e^{-4\pi\varepsilon^2\tau^2\frac{|\xi'|^2}{|\xi|^2}} + \varepsilon^s\right),\end{aligned} \tag{3.7.52}$$

holds true for all $\xi \in \mathbb{R}^n$. Now consider $R \geqslant 1$ (which will be fixed later) and denote by $B_R(0)$ the open ball in $\mathbb{R}^n$ centered at 0 of radius R. For convenience we denote $\widetilde{q} := \chi_{\Omega\cup\Omega^*}(\widetilde{q}_1 - \widetilde{q}_2)$. By Plancherel's theorem , we have

$$\|\widetilde{q}\|^2_{L^2(\mathbb{R}^n)} = \int_{B_R(0)\setminus\{0\}} |\mathcal{F}[\widetilde{q}](\xi)|^2 \, d\xi + \int_{\mathbb{R}^n\setminus B_R(0)} |\mathcal{F}[\widetilde{q}](\xi)|^2 \, d\xi. \tag{3.7.53}$$

From (3.7.52), we get

$$\begin{aligned}
&\int_{B_R(0)\setminus\{0\}} |\mathcal{F}[\widetilde{q}](\xi)|^2 \, d\xi \\
&\leqslant C_1 R^n \Big(\tau^{-2s/(s+2)} + \varepsilon^{2s} + e^{4\tau\kappa} dist\,(C_1^\Gamma, C_2^\Gamma)^2 \\
&\qquad\qquad + \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{2\theta s^2}{(s+2)(n+ns+2s)}} \tau^{2(s+4)/(s+2)}\Big) \\
&\quad + C_1 \int_{B_R(0)\setminus\{0\}} e^{-8\pi\varepsilon^2\tau^2 \frac{|\xi'|^2}{|\xi|^2}} \\
&\leqslant C_2\, R^n \Big(\tau^{-2s/(s+2)} + e^{4\tau\kappa} dist\,(C_1^\Gamma, C_2^\Gamma)^2 + \varepsilon^{2s} + \varepsilon^{-2}\tau^{-2} \\
&\qquad\qquad + \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{2\theta s^2}{(s+2)(n+ns+2s)}} \tau^{2(s+4)/(s+2)}\Big).
\end{aligned}$$

Thus, by equating ε^{2s} and $\varepsilon^{-2}\tau^{-2}$, that is $\varepsilon = \tau^{-1/(s+1)}$, we obtain

$$\begin{aligned}
&\int_{B_R(0)\setminus\{0\}} |\mathcal{F}[\widetilde{q}](\xi)|^2 \, d\xi \\
&\leqslant C_3\, R^n \Big(\tau^{-2s/(s+2)} + e^{4\tau\kappa} dist\,(C_1^\Gamma, C_2^\Gamma)^2 \\
&\qquad\qquad + \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{2\theta s^2}{(s+2)(n+ns+2s)}} \tau^{2(s+4)/(s+2)}\Big).
\end{aligned} \tag{3.7.54}$$

We now turn to estimate the integral term on $\mathbb{R}^n \setminus B_R(0)$ from the right-hand side of (3.7.53). By hypothesis, the functions $\chi_\Omega q_1$ and $\chi_\Omega q_2$ belong to the class of admissible electric potentials $\mathcal{Q}(\Omega, M, s)$. Hence, from Lemma 3.4.1, it follows that $\widetilde{q} \in \mathcal{Q}(\Omega \cup \Omega^*, 2M, s)$. In particular, $\widetilde{q} \in B_s^{2,\infty}(\mathbb{R}^n)$ and $\|\widetilde{q}\|_{B_s^{2,\infty}} \leqslant 2M$. By combining Proposition 10 and Theorem 5, both in [45], we obtain the following chain of embeddings:

$$B_s^{2,\infty}(\mathbb{R}^n) \subset B_{s/2}^{2,2}(\mathbb{R}^n) \subset H^{s/2}(\mathbb{R}^n).$$

Hence, we deduce that $\widetilde{q} \in H^{s/2}(\mathbb{R}^n)$ and its norm in $H^{s/2}(\mathbb{R}^n)$ only depends on a priori bounds for the magnetic and electric potentials. Then, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n \setminus B_R(0)} |\mathcal{F}[\widetilde{q}](\xi)|^2 \, d\xi \\
&= \int_{\mathbb{R}^n \setminus B_R(0)} (1+|\xi|^2)^{-s/2}(1+|\xi|^2)^{s/2} |\mathcal{F}[\widetilde{q}](\xi)|^2 \, d\xi \\
&\leqslant R^{-s} \int_{\mathbb{R}^n \setminus B_R(0)} (1+|\xi|^2)^{s/2} |\mathcal{F}[\widetilde{q}](\xi)|^2 \, d\xi \\
&\leqslant R^{-s} \|\widetilde{q}\|^2_{H^{s/2}(\mathbb{R}^n)} \leqslant C_4 \, R^{-s}.
\end{aligned}
\tag{3.7.55}
$$

Thus, replacing (3.7.54) and (3.7.55) into (3.7.53) we have that there exist two positive constants C_5 and τ_1 such that the estimate

$$
\begin{aligned}
\|\widetilde{q}\|^2_{L^2(\mathbb{R}^n)} \leqslant C_5 \Big(& R^n \tau^{-2s/(s+2)} + R^{-s} + R^n e^{4\tau\kappa} dist\,(C_1^\Gamma, C_2^\Gamma)^2 \\
& + R^n \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{2\theta s^2}{(s+2)(n+ns+2s)}} \tau^{2(s+4)/(s+2)} \Big).
\end{aligned}
\tag{3.7.56}
$$

holds true for all $\tau \geqslant \tau_1$. We consider $R = \tau^{2s/((n+s)(s+2))}$ to equate the two first terms on the left-hand side of the above inequality. Moreover, there exist two positive constants C_6 and τ_2 such that

$$
R^n = \tau^{2ns/((n+s)(s+2))} \leqslant C_6 e^{\tau\kappa}, \quad \tau \geqslant \tau_2.
$$

Hence, from (3.7.54) and (3.7.56) into (3.7.53), we have

$$
\begin{aligned}
\|\widetilde{q}\|^2_{L^2(\mathbb{R}^n)} \leqslant C_7 \Big(& \tau^{-4s/((n+s)(s+2))} + e^{5\tau\kappa} dist\,(C_1^\Gamma, C_2^\Gamma)^2 \\
& + \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{2\theta s^2}{(s+2)(n+ns+2s)}} \tau^{2(2ns+4n+s^2+4s)/((n+s)(s+2))} \Big).
\end{aligned}
\tag{3.7.57}
$$

Now we consider a large enough $\tau > 0$ such that

$$
\left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{2\theta s^2}{(s+2)(n+ns+2s)}} \tau^{2(2ns+4n+s^2+4s)/((n+s)(s+2))}
$$

will be comparable with

$$
\left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{\theta s^2}{(s+2)(n+ns+2s)}}.
$$

Hence, taking $\tau_0 \geqslant \max\{\tau_1, \tau_2\}$ such that $5\kappa\tau_0 \geqslant 1$, it is easy to check that

$$
\tau := \frac{1}{5\kappa} \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{\frac{\theta s^2(n+s)}{2(n+ns+2s)(2ns+4n+s^2+4s)}} \geqslant \tau_0,
\tag{3.7.58}
$$

whenever

$$
dist\,(C_1^\Gamma, C_2^\Gamma) \leqslant e^{-(5\kappa\tau_0)^{\frac{2(n+ns+2s)(2ns+4n+s^2+4s)}{\theta s^2(n+s)}}}.
\tag{3.7.59}
$$

Thus, from (3.7.58) it follows that

$$\tau^{-4s/((n+s)(s+2))} \leqslant C_8 \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{2\theta s^3}{(s+2)(n+ns+2s)(2ns+4n+s^2+4s)}}. \tag{3.7.60}$$

From (3.7.58), we have

$$e^{5\tau\kappa} dist\,(C_1^\Gamma, C_2^\Gamma)^2 = e^{5\tau\kappa - 2(5\tau\kappa)^{\frac{2(n+ns+2s)(2ns+4n+s^2+4s)}{\theta s^2(n+s)}}} \leqslant e^{-5\tau\kappa},$$

where we have used that $dist\,(C_1^\Gamma, C_2^\Gamma) \leqslant e^{-1}$. This fact can be easily deduced from (3.7.59) and $5\kappa\tau \geqslant 1$. Then

$$e^{5\tau\kappa} dist\,(C_1^\Gamma, C_2^\Gamma)^2 \leqslant \frac{1}{5\tau\kappa} = \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{\frac{-\theta s^2(n+s)}{2(n+ns+2s)(2ns+4n+s^2+4s)}}. \tag{3.7.61}$$

By construction, the last term on the right-hand side of (3.7.57) satisfies

$$\begin{aligned} &\left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{2\theta s^2}{(s+2)(n+ns+2s)}} \tau^{4(ns+2n+s+4)/((n+2)(s+2))} \\ &\qquad \leqslant C_{10} \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{\theta s^2}{(s+2)(n+ns+2s)}}. \end{aligned} \tag{3.7.62}$$

By replacing (3.7.60)-(3.7.62) into (3.7.57), we obtain

$$\|\widetilde{q}\|_{L^2(\mathbb{R}^n)} \leqslant C_{11} \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{\theta s^3}{(s+2)(n+ns+2s)(2ns+4n+s^2+4s)}}. \tag{3.7.63}$$

Now since $n \geqslant 3$, we have

$$n + \frac{n}{2} + 1 \leqslant 2n, \qquad 5n + \frac{9}{4} \leqslant 6n,$$

an since $s \in (0, 1/2)$, we get

$$\begin{aligned} &\frac{\theta s^3}{(s+2)(n+ns+2s)(2ns+4n+s^2+4s)} \\ &\qquad \geqslant \frac{2\theta s^3}{5(n+\frac{n}{2}+1)(5n+\frac{9}{4})} \geqslant \frac{\theta s^3}{30n^2}. \end{aligned}$$

By replacing this inequality into (3.7.63), we get

$$\|\widetilde{q}\|_{L^2(\mathbb{R}^n)} \leqslant C_{11} \left|\log dist\,(C_1^\Gamma, C_2^\Gamma)\right|^{-\frac{\theta s^3}{30n^2}}.$$

Thus, by taking $\lambda = \theta/30$, C as follows

$$C = \max\left\{ (5\kappa\tau_0)^{\frac{2(n+ns+2s)(2ns+4n+s^2+4s)}{\theta s^2(n+s)}}, C_{11} \right\}$$

and also taking into account that

$$\|q_1 - q_2\|_{L^2(\Omega)} \leqslant \|\widetilde{q}\|_{L^2(\mathbb{R}^n)},$$

we conclude the proof.

3.8 Identifiability for the magnetic fields and the electric potentials

3.8.1 Proof of Theorem 1.2.1

We only give the main ideas to prove the identifiability for the magnetic field and electric potential since it is just the qualitative version of what we have proved in the previous sections. We consider ρ_1 and ρ_2 given by (3.4.12). Now let $U_1, U_2 \in H^1(\Omega)$ satisfying $\mathcal{L}_{A_1,q_1}U_1 = 0$ with $U_1|_{\Gamma_0} = 0$ and $\mathcal{L}_{\overline{A_2},\overline{q_2}}U_2 = 0$ with $U_2|_{\Gamma_0} = 0$. The existence of such functions are given by Proposition 3.4.3, except that we replaced the estimates (3.4.4)-(3.4.9) by the estimates from Remark 3.3.3, (3.3.17)-(3.3.19). Thus, since $C^{\Gamma}_{A_1,q_1} = C^{\Gamma}_{A_2,q_2}$, Corollary 3.2.2 ensures that

$$\int_\Omega (A_1 - A_2) \cdot \left(DU_1\overline{U_2} + U_1\overline{DU_2}\right) + \left(A_1^2 - A_2^2 + q_1 - q_2\right) U_1\overline{U_2} = 0.$$

From this integral identity we can prove the identifiability for the magnetic potentials, following the proof of the Proposition 3.5.1, applying the Riemann–Lebesgue lemma to the function $\chi_\Omega(A_1 - A_2)e^{\Phi_1^\sharp + \overline{\Phi_2^{\sharp *}}}$ to estimate (3.5.19)-(3.5.20) and taking into account Lemma 3.5.4 in order to remove the term $e^{\Phi_1 + \overline{\Phi_2}}$ on the left-hand side of (3.5.31). At this point, since the Fourier transform is analytic, Proposition 3.5.1 and (3.6.1) give us the following equality in the sense of the distributions in $\mathbb{R}^n$:

$$d(\chi_{\Omega\cup\Omega^*}\widetilde{A_1}) = d(\chi_{\Omega\cup\Omega^*}\widetilde{A_2}),$$

which implies that $dA_1 = dA_2$ in Ω.

The proof of the identifiability for the electric potential is as follows. We consider the Hodge decomposition for $\chi_{\Omega\cup\Omega^*}(\widetilde{A_1} - \widetilde{A_2})$ in a ball B satisfying $\Omega \cup \Omega^* \subset\subset B$. We also take into account the estimates from Remark 3.3.3 and the Riemann–Lebesgue lemma applied to the function $\chi_{\Omega\cup\Omega^*}(\widetilde{q_1} - \widetilde{q_2})$. Finally, since the Fourier transform is analytic, Proposition 3.7.3 and (3.7.52) imply that

$$\chi_{\Omega\cup\Omega^*}\widetilde{q_1} = \chi_{\Omega\cup\Omega^*}\widetilde{q_2},$$

so we have $q_1 = q_2$ in Ω.

Appendix A

A.1 Limiting Carleman weight - LCW

As we have already mentioned, the construction of special solutions for the magnetic Schrödinger equation $\mathcal{L}_{A,q}u = 0$ is closely related to the deduction of a Carleman estimate for the Laplace operator. Roughly speaking, the main idea in a Carleman estimate is to obtain first an estimate for the Laplace operator, of the form:

$$\|u\|_H \leqslant C \left\| \tau^{-2} e^{\tau\varphi} \Delta(e^{-\tau\varphi} u) \right\|_W \pm \text{boundary error terms}, \tag{A.1.1}$$

where C is a constant only depending on the a priori assumptions over the magnetic potential A, the electric potential q, the domain Ω and the dimension n. The spaces H and W denote suitable Hilbert spaces. The real-valued function φ is smooth enough. For instance, for full data cases (the boundary error terms are equal to zero), Carleman estimates were deduced by considering $\varphi(x) = \xi \cdot x$ with $\xi \in \mathbb{S}^n$. The key to deduce such Carleman estimates were to prove that the commutator $\tau^{-2}\left[-\Delta, e^{\tau\xi\cdot x}\right] u$ behaves well in the following sense: the terms coming from the commutator can be absorbed by terms on the left-hand side of (A.1.1). This can be seen as a direct consequence of the properties of $\varphi(x) = \xi \cdot x$. These facts were noticed by Kenig, Sjostrand and Uhlmann [26] when working on the identifiability issue when Ω is illuminated from a point x_0 and for partial data case. In their work, they used $\varphi(x) = \log|x - x_0|$ and introduced the notion of Limiting Carleman weights.

Definition A.1.1. We say that a smooth real-valued function φ is a **Limiting Carleman weight**, LCW for short, in an open bounded set $\Omega \subset \mathbb{R}^n$ if it has nonvanishing gradient and satisfies pointwise in Ω

$$\left\langle \varphi'' \nabla\varphi, \nabla\varphi \right\rangle + \left\langle \varphi'' \zeta, \zeta \right\rangle = 0, \tag{A.1.2}$$

whenever $|\zeta| = |\nabla\varphi|$ and $\nabla\varphi \cdot \zeta = 0$.

It was shown in [16] that there exist only six LCWs for open bounded sets in $\mathbb{R}^n$. It is easy to see that $\varphi(x) = \xi \cdot x$ and $\varphi(x) = \log|x - x_0|$ satisfy the definition of LCW. Different kinds of Carleman estimates with or without boundary error terms, were obtained by several authors, see for example [5], [17] and [26].

Appendix B

In this part of the appendix we prove an estimate, on a suitable space, for the solutions Φ of the following transport equation in $\mathbb{R}^n$:

$$(\xi + i\zeta) \cdot \nabla\Phi = F \tag{B.0.1}$$

where F is a smooth enough fixed function. Here $\xi \in S^{n-1}$ and $\zeta \in S^{n-1}$ with $\xi \cdot \zeta = 0$. The main result of this appendix is Proposition B.3.1.

B.1 Cauchy transform

Let F be a function on $\mathbb{R}^n$. The Cauchy transform on $\mathbb{R}^n$ of F is defined by

$$(C_{\xi+i\zeta}\, F)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{y_1 + iy_2} F(x - y_1\xi - y_2\zeta) dy_1 dy_2,$$

whenever the integral exists. This transform allows us to obtain a solution for the the equation (B.0.1). More precisely, we have the following lemma proved by Sun [46]. See also [41].

Lemma B.1.1. *Let $F \in W^{k,\infty}(\mathbb{R}^n)$, $k \geqslant 0$, with $\operatorname{supp}(F) \subset B_R(0)$, $R > 0$. Then $\Phi = C_{\xi+i\zeta}\, F \in W^{k,\infty}(\mathbb{R}^n)$ solves (B.0.1), and satisfies*

$$\|C_{\xi+i\zeta}\, F\|_{W^{k,\infty}(\mathbb{R}^n)} = \|\Phi\|_{W^{k,\infty}(\mathbb{R}^n)} \leqslant C\, \|F\|_{W^{k,\infty}(\mathbb{R}^n)}\,,$$

where $C > 0$ only depends on R. Moreover, if $F \in C(\mathbb{R}^n)$ and has compact support then $\Phi \in C(\mathbb{R}^n)$.

This lemma tells us that the solutions of the transport equation (B.0.1) inherit the same smoothness of the function F. This fact is more than enough to perform all arguments in Chapter 2, except the construction of a function $b \in H^1(\Omega)$ satisfying (C.1.59). Since we would like to have a function b expanded until the second order, see (C.1.60), we have to prove a more refined estimate than given in Lemma B.1.1.

B.2 The Beurling transform

There is another transform associated with the Cauchy transform, the so-called Beurling transform $S_{\xi-i\zeta}$. It is defined on $\mathbb{R}^n$ by

$$(S_{\xi-i\zeta}\, G)(x) = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{1}{(y_1+iy_2)^2} G(x - y_1\xi - y_2\zeta)dy_1dy_2.$$

Note that $1/(y_1+iy_2)^2$ is not integrable on $\mathbb{R}^2$. Hence, the above integral has to be seen in terms of its principal value integral:

$$(S_{\xi-i\zeta}\, G)(x) = \frac{1}{\pi}\lim_{s\to 0}\int_{|y_1+iy_2|>s} \frac{1}{(y_1+iy_2)^2} G(x - y_1\xi - y_2\zeta)dy_1dy_2.$$

The following lemma can be found in [2], see Theorem 4.3.10 therein.

Lemma B.2.1. *Assume that $F \in L^2(\mathbb{R}^n)$. Then we have the following identities in the sense of the distributions:*

$$[(\xi + i\zeta)\cdot\nabla]\,(C_{\xi+i\zeta}\, F) = F, \tag{B.2.1}$$

and

$$[(\xi - i\zeta)\cdot\nabla]\,(C_{\xi+i\zeta}\, F) = S_{\xi-i\zeta}\, F. \tag{B.2.2}$$

Moreover, $S_{\xi-i\zeta} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded operator for all $1 < p < \infty$. In particular, for $p = 2$ is an isometry. Finally, if $G \in W^{m,2}(\mathbb{R}^n)$ with $m \in \mathbb{N}$, then

$$\partial^\alpha(S_{\xi-i\zeta}\, G) = S_{\xi-i\zeta}\,(\partial^\alpha G), \quad |\alpha| \leqslant m.$$

In order to obtain a more refined estimate for the Cauchy transform than given in Lemma B.1.1, we introduce the so-called Besov spaces.

Definition B.2.2. Given $\gamma \in (0,1)$, $p \in (1,\infty)$ and $m \in \mathbb{N}$, we define the **Besov space** $B^{p,\infty}_{m+\gamma}(\mathbb{R}^n)$ as the space consisting of all functions $f \in W^{m,\infty}(\mathbb{R}^n)$ endowed with the norm

$$\|f\|_{B^{p,\infty}_{m+\gamma}(\mathbb{R}^n)} = \|f\|_{W^{m,\infty}(\mathbb{R}^n)} + \sum_{|\alpha|=m}\sup_{y\in\mathbb{R}^n\setminus\{0\}} \frac{\|\partial^\alpha\,[f(\cdot+2y) - f(\cdot+y) + f(\cdot)]\|_{L^p(\mathbb{R}^n)}}{|y|^\gamma}. \tag{B.2.3}$$

Remark B.2.3. *This definition was taken from Triebel's book [49], see Section 2.2.2 therein. We mention that when $m = 0$ and $p = 2$ the above norm is equivalent with the norm defined in (1.2.6). For this fact, see for instance Proposition $8'$ in Chapter 5 of [45].*

Lemma B.2.4. *Let $\gamma \in (0,1)$, p be a real number with $1 < p < \infty$ and $m \in \mathbb{N}$. Let $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ be a bounded operator satisfying the following property*

$$\partial^\alpha(T\, G) = T(\partial^\alpha\, G), \quad |\alpha| \leqslant m.$$

Then there exists $C = C(p) > 0$ such that

$$\|TG\|_{B^{p,\infty}_{m+\gamma}(\mathbb{R}^n)} \leqslant C\,\|G\|_{B^{p,\infty}_{m+\gamma}(\mathbb{R}^n)}\,.$$

Proof. The proof is an immediate consequence of the commutator property $\partial^\alpha(T\,G) = T(\partial^\alpha\,G)$ and the definition of the norm on the Besov space $B^{p,\infty}_{m+\gamma}(\mathbb{R}^n)$. $\square$

Corollary B.2.5. *Let $\gamma \in (0,1)$, p be a real number with $1 < p < \infty$ and $m \in \mathbb{N}$. Then there exists $C > 0$ such that the following estimate*

$$\|S_{\xi-i\zeta}\,G\|_{B^{p,\infty}_{m+\gamma}(\mathbb{R}^n)} \leqslant C\,\|G\|_{B^{p,\infty}_{m+\gamma}(\mathbb{R}^n)}\,,$$

holds true for all $G \in B^{p,\infty}_{m+\gamma}(\mathbb{R}^n) \cap W^{m,2}(\mathbb{R}^n)$.

Proof. The proof is immediate because from Lemma B.2.1, the operator $S_{\xi-i\zeta}$ satisfies all conditions from Lemma B.2.4. $\square$

We end this section with the following embeddings.

Lemma B.2.6. *Let $\gamma \in (0,1)$ and $m \in \mathbb{N}$. Consider $p > 1$ such that $n/p < \gamma$. Then there exists $C > 0$ such that*

$$\|G\|_{C^{m,\gamma-n/p}} \leqslant C\,\|G\|_{B^{p,\infty}_{m+\gamma}(\mathbb{R}^n)}\,, \quad G \in B^{p,\infty}_{m+\gamma}(\mathbb{R}^n). \tag{B.2.4}$$

If G has compact support then we have

$$\|G\|_{B^{p,\infty}_{m+\gamma}(\mathbb{R}^n)} \leqslant C\,\|G\|_{C^{m,\gamma}(\mathbb{R}^n)}\,, \quad G \in C^{m,\gamma}(\mathbb{R}^n). \tag{B.2.5}$$

Proof. The first embedding can be found in [49], see Remark 2 in Section 2.7.1 therein. We now prove the second estimate. It was proved in Section 2.5.12 in [49], that the norm $\|\ \|_{H_M}$, where M is any integer with $M > m + \gamma$, defined by

$$\|f\|_{H_M} = \|f\|_{L^p(\mathbb{R}^n)} + \sup_{0<|y|\leqslant 1} \frac{\left\|\Delta^M_y f\right\|_{L^p(\mathbb{R}^n)}}{|y|^s}, \quad M > m+\gamma \tag{B.2.6}$$

and the norm $\|\ \|_{B^{p,\infty}_{m+\gamma}(\mathbb{R}^n)}$ defined by (B.2.3), are equivalent norm in the Besov space $B^{p,\infty}_{m+\gamma}(\mathbb{R}^n)$. Here, for a fixed $y \in \mathbb{R}^n$, we have denoted $(\Delta^1_y\,f)(x) = f(x+y) - f(x)$ and recursively for any integer $l \geqslant 2$, $(\Delta^l_y\,f)(x) = \Delta^1_y(\Delta^{l-1}_y f)(x)$. Observe that this equivalent norm only consider the differences until the M-th order of the function f while the original norm consider the derivatives until the m-th order. The other difference is that the supp is taken on the unit ball minus the origin. Combining this fact, the supporting compactness of G and taking into account the equivalent norm H_M, the second embedding is easily followed. $\square$

B.3 An estimate associated to the transport equation

We are now in position to prove the main result of this part of the appendix.

Proposition B.3.1. *Let $\xi \in S^{n-1}$ and consider $\zeta \in S^{n-1}$ such that $\xi \cdot \zeta = 0$. Let $\gamma \in (0,1)$ and $m \in \mathbb{N}$. Let $F \in C^{m,\gamma}(\mathbb{R}^n)$ with $\operatorname{supp}(F) \subset B_R(0)$, $R > 0$. Then there exists a small constant $\underline{\gamma}$ with $0 < \underline{\gamma} < \gamma$ such that the function $\Phi := C_{\xi+i\zeta} F \in C^{m+1,\underline{\gamma}}(\mathbb{R}^n)$ solves (B.0.1) in $\mathbb{R}^n$, and satisfies*

$$\|C_{\xi+i\zeta} F\|_{C^{m+1,\underline{\gamma}}(\mathbb{R}^n)} = \|\Phi\|_{C^{m+1,\underline{\gamma}}(\mathbb{R}^n)} \leqslant C \|F\|_{C^{m,\gamma}(\mathbb{R}^n)},$$

where $C > 0$ only depends on R.

Proof. We start by verifying that Φ is a solution of (B.0.1). Since F has compact support, it follows that $F \in W^{m,2}(\mathbb{R}^n)$. Hence, identity (B.2.1) in Lemma B.2.1, shows that in fact Φ is a solution. Now let $p > 1$ large enough such that $\underline{\gamma} = \gamma - n/p > 0$. Then

$$\begin{aligned}
&\text{(B.3.1)} && \|(\xi - i\zeta) \cdot \nabla\Phi\|_{C^{m,\underline{\gamma}}(\mathbb{R}^n)} = \|(\xi - i\zeta) \cdot \nabla\Phi\|_{C^{m,\gamma-n/p}(\mathbb{R}^n)} \\
&\text{(B.3.2)} && \qquad = \|[(\xi - i\zeta) \cdot \nabla] (C_{\xi+i\zeta}F)\|_{C^{m,\gamma-n/p}(\mathbb{R}^n)} \\
&\text{(B.3.3)} && \qquad = \|S_{\xi-i\zeta} F\|_{C^{m,\gamma-n/p}(\mathbb{R}^n)} \\
&\text{(B.3.4)} && \qquad \leqslant C_1 \|S_{\xi-i\zeta} F\|_{B^{p,\infty}_{m+\gamma}(\mathbb{R}^n)} \\
&\text{(B.3.5)} && \qquad \leqslant C_2 \|F\|_{B^{p,\infty}_{m+\gamma}(\mathbb{R}^n)} \\
&\text{(B.3.6)} && \qquad \leqslant C_3 \|F\|_{C^{m,\gamma}(\mathbb{R}^n)},
\end{aligned}$$

where in the above steps (B.3.3)-(B.3.6) we have used respectively (B.2.2), (B.2.4), Corollary B.2.5 and (B.2.5). On the other hand, from (B.2.1) it is easy to see that

$$\begin{aligned}
\|(\xi + i\zeta) \cdot \nabla\Phi\|_{C^{m,\underline{\gamma}}(\mathbb{R}^n)} &= \|[(\xi + i\zeta) \cdot \nabla] (C_{\xi+i\zeta} F)\|_{C^{m,\underline{\gamma}}(\mathbb{R}^n)} \\
&= \|F\|_{C^{m,\underline{\gamma}}(\mathbb{R}^n)} \leqslant C_4 \|F\|_{C^{m,\gamma}(\mathbb{R}^n)}
\end{aligned}$$

Combining the two above inequalities, we get the following estimate

$$\|\mu \cdot \nabla\Phi\|_{C^{m,\underline{\gamma}}(\mathbb{R}^n)} \leqslant C_5 |\mu| \|F\|_{C^{m,\gamma}(\mathbb{R}^n)},$$

holds true for all $\mu \in \operatorname{span}\{\xi, \zeta\}$. We conclude the proof by taking in this inequality the family of canonical vectors in $\mathbb{R}^n$: $\mu = e_j$ with $j = 1, \ldots, n$. $\square$

Appendix C

C.1 Remarks on CGO solutions

The main goal of this section is the construction of CGO solutions in Ω for the equation $\mathcal{L}_{A,q}u = 0$ with the vanishing condition on a compact subset of $\partial\Omega_{-,\xi}$. More precisely, we will prove the following existence result.

Theorem C.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Let $\xi, \zeta \in S^{n-1}$ be a pair of orthonormal vectors and let E be a compact subset of $\partial\Omega_{-,\xi}$. Consider $\gamma \in (0,1)$. If $A \in C^{2,\gamma}(\overline{\Omega};\mathbb{R}^n)$ and $q \in L^\infty(\Omega)$, then there exist three positive constants τ_0, C (both depending on $n, \Omega, \|A\|_{C^{2,\gamma}}, \|q\|_{L^\infty}$) and $\underline{\gamma}$ with $0 < \underline{\gamma} < \gamma$ such that the equation*

$$\begin{cases} \mathcal{L}_{A,q}\, u = 0 & \textit{in } \Omega \\ u|_E = 0 \end{cases}$$

has a solution $u \in H^1(\Omega)$ of the form

$$u = e^{\tau(\xi\cdot x + i\zeta\cdot x)}\left(e^{\Phi} + r\right) - e^{\tau l} b,$$

with the following properties:

(i) The function $\Phi \in C^{3,\underline{\gamma}}(\Omega)$ satisfies in Ω

$$(\xi + i\zeta)\cdot\nabla\Phi + i(\xi + i\zeta)\cdot A = 0, \tag{C.1.1}$$

$$\|\Phi\|_{W^{\alpha,\infty}(\Omega)} \leqslant C\,\|A\|_{C^{\alpha}(\Omega)}\ ,\ |\alpha| \leqslant 2. \tag{C.1.2}$$

and

$$\|\Phi\|_{C^{3,\underline{\gamma}}(\Omega)} \leqslant C\,\|A\|_{C^{2,\gamma}(\Omega)}\,. \tag{C.1.3}$$

(ii) The function l depends on the a priori bounds of A and q, and satisfies

$$\Re l(x) = \xi\cdot x - k(x), \tag{C.1.4}$$

where $k(x) \simeq dist(x, E)$ in G, a neighborhood of E on $\mathbb{R}^n$.

(iii) The function b belongs to $C^{1,\underline{\gamma}}(\Omega)$ with $\operatorname{supp} b \subset G$; and it depends on the a priori bounds of A and q.

(iv) Finally, $r \in H^1(\Omega)$ satisfies $r|_E = 0$ and for all $\tau \geqslant \tau_0$ the following estimates hold true

$$\begin{aligned} \|\partial^\alpha r\|_{L^2(\Omega)} &\leqslant C\tau^{|\alpha|-1}, |\alpha| \leqslant 1, \\ \|r\|_{L^2(\partial\Omega)} &\leqslant C\tau^{-1/2}. \end{aligned} \tag{C.1.5}$$

Moreover, we have

$$\|l\|_{H^1(\Omega)} \leqslant C, \quad \|b\|_{H^1(\Omega)} \leqslant C \tag{C.1.6}$$

and

$$\left\|e^{-\tau k}\right\|_{L^2(\Omega)} \leqslant C\tau^{-1/2}, \quad \left\|e^{-\tau k}\right\|_{L^\infty(\Omega)} \leqslant C. \tag{C.1.7}$$

The version of this theorem in the context of illuminating Ω from a point $x_0 \in \mathbb{R}^n$, was proved by Chung by considering a logarithmic limiting Carleman weight $\log|x - x_0|$, see Proposition 9.2 in [14]. As it is well known, the main ingredient in the construction of special solutions is a suitable Carleman estimate. In this way, a novel Carleman estimate with logarithmic weight was derived by Chung, see Theorem 1.4 in [14]. The key result to prove the aforementioned theorem was Proposition 3.1 in [14]. Actually, Chung dedicated several sections in his article employing elegant and original arguments to prove this proposition. For us, this proposition is the heart of Chung's paper. Thereby, following Chung's ideas, we would like to have the analogous of Proposition 3.1 in [14] with the linear limiting Carleman weight $\xi \cdot x$ instead of the logarithmic. Fortunately, we have noticed that hidden within the proof of the aforementioned proposition, there is an implicit result which can be used to deduce a suitable Carleman estimate for our case. We collect such implicit results in Lemma C.1.2. Before stating it, we introduce some notations and assumptions. For a large parameter $\tau > 0$ and a limiting Carleman weight φ, we set

$$\mathcal{L}_{A,q,\varphi} = \tau^{-2} e^{\tau\varphi} \mathcal{L}_{A,q} e^{-\tau\varphi} \tag{C.1.8}$$

and for $\varepsilon > 0$ small enough, we set

$$\mathcal{L}_{A,q,\varphi,\varepsilon} = e^{\varphi^2/2\varepsilon} \mathcal{L}_{A,q,\varphi} e^{-\varphi^2/2\varepsilon}. \tag{C.1.9}$$

For a subset $V \subset \mathbb{R}^n$, we shall denote by $H^1_{scl}(V)$ the H^1-Sobolev space, with semiclassical parameter τ^{-1}, equipped with the norm:

$$\|u\|_{H^1_{scl}(V)} = \|u\|_{L^2(V)} + \left\|\tau^{-1}\nabla u\right\|_{L^2(V)}$$

and its dual space by $H^{-1}_{scl}(V)$, whose norm is defined by

$$\|u\|_{H^{-1}_{scl}(V)} = \sup_{\psi \in C_0^\infty(V)\setminus\{0\}} \frac{|\langle u, \psi\rangle_V|}{\|\psi\|_{H^1_{scl}(V)}}, \tag{C.1.10}$$

where $\langle\cdot,\cdot\rangle_V$ denotes the distribution duality in V. Also we denote by $x = (x', x_n) \in \mathbb{R}^n$ where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$.

Lemma C.1.2. *Let $\mathcal{L}_{\tau,\varepsilon}$ be a second-order semiclassical operator on*

$$\mathbb{R}^n_{+1} = \left\{(\theta, r) \in \mathbb{R}^{n-1} \times \mathbb{R} \,:\, r \geqslant 1\right\}$$

of the form

$$\mathcal{L}_{\tau,\varepsilon} = \left(1 + |F|^2\right) \tau^{-2}\partial_r^2 - \frac{2}{r}\left(a + G\cdot\tau^{-1}\nabla_\theta\right)\tau^{-1}\partial_r + \frac{1}{r^2}\left(a^2 + \tau^{-1}L_\theta\right), \tag{C.1.11}$$

where F and G are smooth vector fields, a is a smooth real-valued function and L_θ is a second-order differential operator of the form

$$L_\theta = a_1\partial^2_{\theta_1} + a_2\partial^2_{\theta_2} + \ldots + a_{n-1}\partial^2_{\theta_{n-1}} + \textit{first and zero order terms},$$

where $\theta = (\theta_1, \ldots, \theta_{n-1})$ and $(a_j)_{j=1}^{n-1}$ are smooth real valued functions. Let U and U_2 be two bounded open sets on $\mathbb{R}^n_{+1} \setminus \{(0,1)\}$ with smooth boundaries such that $U \subsetneq U_2$ and $\emptyset \neq \partial U \cap \partial U_2 \subset \partial\mathbb{R}^n_{+1}$. Assume that there exist three positive constants C, δ, τ_0 and $K \in \mathbb{R}^n$ such that:
(i). The operator $\mathcal{L}_{\tau,\varepsilon}$ satisfies the following estimate

$$\frac{\tau^{-1}}{\sqrt{\varepsilon}}\,\|w\|_{H^1_{scl}(U_2)} \leqslant C\,\|\mathcal{L}_{\tau,\varepsilon}w\|_{L^2(U_2)}, \tag{C.1.12}$$

for all $w \in C_0^\infty(U_2)$ and for all $\tau \geqslant \tau_0$.
(ii). The coefficients of the operator $\mathcal{L}_{\tau,\varepsilon}$ satisfy

$$|G - K| \leqslant \delta, \quad \textit{in}\, U_2, \tag{C.1.13}$$

$$||F| - |K|| \leqslant \delta, \quad \textit{in}\, U_2, \tag{C.1.14}$$

$$|a_j - 1| \leqslant \delta, \quad j = 1, \ldots, n-1, \quad \textit{in}\, U_2 \tag{C.1.15}$$

and

$$|a - 1| \leqslant C\,\frac{\tau^{-1}}{\varepsilon}, \quad \textit{in}\, U_2, \tag{C.1.16}$$

for all $\tau \geqslant \tau_0$. Then there exist two positive constants C_1 and τ_1 such that

$$\frac{\tau^{-1}}{\sqrt{\varepsilon}}\,\|w\|_{L^2(U)} \leqslant C_1\,\|\mathcal{L}_{\tau,\varepsilon}w\|_{H^{-1}_{scl}(\mathbb{R}^n_{+1})}, \tag{C.1.17}$$

holds true for all $w \in C_0^\infty(U)$ and for all $\tau \geqslant \tau_1$.

Remark C.1.3. This result shows that it is possible to transfer the information from an $H^1(U_2)$-$L^2(U_2)$ estimate of the form (C.1.12) into an $L^2(U)$-$H^{-1}(\mathbb{R}^n_{+1})$ estimate of the form (C.1.17). But this is not new in Operator Theory: it can be done by a relatively standard commutator method. Unfortunately, in general, the commutator method does not preserve the support of the functions. The novelty of this result is that not only included the case when $\partial U \cap \partial \mathbb{R}^n_{+1} = \emptyset$ but also the case when $\partial U \cap \partial \mathbb{R}^n_{+1} \neq \emptyset$. The latter case is more delicate and difficult to face, because we have to keep through all the computations the support constraint on $\partial \mathbb{R}^n_{+1}$. To deal with these difficulties and for technical reasons, Chung first worked in a slightly larger domain than U, namely U_2, and also constructed nice operators on $\mathbb{R}^n_{+1}$ whose main property is the preservation of the support constraint along $\partial \mathbb{R}^n_{+1}$, see section 5 in [14].

Remark C.1.4. This result was implicitly stated in [14]. It follows by combining the conditions (3.4)-(3.8) and lemmas 4.1 and 4.2, all in [14], see sections 2 and 3 therein. Now we make the analogies between the aforementioned conditions from [14] and the conditions stated in Lemma C.1.2 here. The condition (C.1.12) is (3.4) in [14]. The conditions (C.1.13)-(C.1.15) here are (3.6)-(3.8) in [14], respectively. The condition over a, see (C.1.16), can be deduced from the definition of its analogous in Lemma 3.2 in [14]. Finally, the form of the operator $\mathcal{L}_{\tau,\varepsilon}$ is given by (3.5) in [14].

To avoid long computations, from now on and unless otherwise indicated, we assume that $\xi = e_n$ (the n-th canonical unit vector in $\mathbb{R}^n$) and so our linear limiting Carleman weight will be $\varphi(x) = x_n$. For an arbitrary $\xi \in \mathbb{S}^n$, it is enough to make a change of coordinates given by a rotational transformation T in $\mathbb{R}^n$ such that $T(\xi) = e_n$.

Proposition C.1.5. *Consider the linear limiting Carleman weight $\varphi(x) = x_n$. Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be a smooth function such that*

$$E \subset \left\{ x = (x', x_n) \,:\, x_n = f(x') \right\}$$

and $\Omega \subset \Xi_f$, where Ξ_f is defined by

$$\Xi_f := \left\{ x = (x', x_n) \,:\, x_n \geqslant f(x') \right\}.$$

Assume that there exist $K \in \mathbb{R}^n$ and $\delta > 0$ such that

$$|\nabla_{x'} f - K| < \delta. \tag{C.1.18}$$

Then there exists a constant $C > 0$ (depending on K, δ, f and Ω) such that for all $w \in C_0^\infty(\Omega)$, we have

$$\frac{\tau^{-1}}{\sqrt{\varepsilon}} \|w\|_{L^2(\Omega)} \leqslant C \|\mathcal{L}_{0,0,\varphi,\varepsilon} w\|_{H^{-1}_{scl}(\Xi_f)}. \tag{C.1.19}$$

The proof of this proposition will be done in several steps, by making a suitable change of variables in order to be in the position to apply Lemma C.1.2. The first step wil be to flatten out the subset of the boundary E into the hyperplane $\{(x', x_n) \in \mathbb{R}^n \,:\, x_n = 0\}$.

Lemma C.1.6. *Consider the following change of variables Υ_1 on $\mathbb{R}^n_0 = \{(x', x) \in \mathbb{R}^n \,:\, x_n \geqslant 0\}$, defined by*

$$\Upsilon_1(x', x_n) = (x', x_n - f(x')).$$

Then the operator $\mathcal{L}_{0,0,\varphi,\varepsilon}$ in Υ_1-coordinates has the form:

$$\begin{aligned}\mathcal{L}_{0,0,\varphi,\varepsilon,\Upsilon_1} &= (1+|\nabla_{x'}f|^2)\tau^{-2}\partial_{x_n}^2 - 2(\alpha_{\Upsilon_1} - \nabla_{x'}f\cdot\tau^{-1}\nabla_{x'})\tau^{-1}\partial_{x_n}\\ &\quad + \alpha_{\Upsilon_1}^2 + \tau^{-2}\Delta_{x'} + (\tau^{-1}\Delta_{x'}f)\tau^{-1}\partial_{x_n} - \frac{\tau^{-2}}{\varepsilon},\end{aligned} \tag{C.1.20}$$

where

$$\alpha_{\Upsilon_1} = 1 + \frac{\tau^{-1}}{\varepsilon}(x_n + f(x')).$$

Proof. We start by computing the full form of the operator $\mathcal{L}_{0,0,\varphi,\varepsilon}$ in the original coordinates (x', x_n). From (C.1.8)-(C.1.9), we have

$$\mathcal{L}_{0,0,\varphi,\varepsilon} = \tau^{-2}e^{\tau\varphi+\frac{\varphi^2}{2\varepsilon}}\Delta\, e^{-\tau\varphi-\frac{\varphi^2}{2\varepsilon}}. \tag{C.1.21}$$

For a real-valued function ρ and a smooth function w on $\mathbb{R}^n_0$, an immediate computation give us

$$e^{\rho}\Delta(e^{-\rho}w) = (\nabla\rho\cdot\nabla\rho - \Delta\rho)\, w - 2\nabla\rho\cdot\nabla w + \Delta w.$$

Applying this identity with $\rho = \tau\varphi + \varphi^2/2\varepsilon$ and taking into account the identities

$$\begin{aligned}\nabla\rho &= \nabla\left(\tau\varphi + \varphi^2/2\varepsilon\right) = \tau\nabla\varphi + \frac{1}{\varepsilon}\varphi\nabla\varphi = \tau\, e_n\left(1 + \frac{\tau^{-1}}{\varepsilon}x_n\right),\\ \Delta\rho &= \nabla\cdot\nabla\rho = \nabla\cdot\left(\tau\nabla\varphi + \frac{1}{\varepsilon}\varphi\nabla\varphi\right) = \tau\Delta\varphi + \frac{1}{\varepsilon}\left(\nabla\varphi\cdot\nabla\varphi + \varphi\Delta\varphi\right) = \frac{1}{\varepsilon},\end{aligned}$$

we have

$$\begin{aligned}\mathcal{L}_{0,0,\varphi,\varepsilon}w(x) = \Bigg[&\tau^{-2}\Delta_x - 2\left(1+\frac{\tau^{-1}}{\varepsilon}x_n\right)\tau^{-1}\partial_{x_n}\\ &+ \left(1+\frac{\tau^{-1}}{\varepsilon}x_n\right)^2 - \frac{\tau^{-2}}{\varepsilon}\Bigg]\, w(x).\end{aligned} \tag{C.1.22}$$

Now let $\widetilde{w}$ be a smooth function on $\Upsilon_1(\mathbb{R}^n_0)$ and consider w be a smooth function on $\mathbb{R}^n_0$ such that $\widetilde{w} = w\circ\Upsilon_1^{-1}$. The task now is to compute $\mathcal{L}_{0,0,\varphi,\varepsilon}\widetilde{w}$, thus we have

$$\mathcal{L}_{0,0,\varphi,\varepsilon}\widetilde{w}(x', x_n) = \mathcal{L}_{0,0,\varphi,\varepsilon}\left[w(x', x_n + f(x'))\right]. \tag{C.1.23}$$

For $j = 1,\dots,n-1$, we set $x' = (x_1,\dots,x_{n-1})$ and by the chain rule we get

$$\begin{aligned}\partial_{x_j}\left(w(x', x_n + f(x'))\right) &= \left(\partial_{x_j}w\right)\left(x', x_n + f(x')\right)\\ &\quad + \left[\left(\partial_{x_n}w\right)\left(x', x_n + f(x')\right)\right]\cdot\left[\partial_{x_j}f(x')\right]\\ &= \partial_{x_j}\widetilde{w}(x', x_n) + \left[\partial_{x_n}\widetilde{w}(x', x_n)\right]\cdot\left[\partial_{x_j}f(x')\right],\end{aligned}$$

which implies

$$
\begin{aligned}
\partial^2_{x_j}\left(w(x',x_n+f(x'))\right) &= \left(\partial^2_{x_j} w\right)\left(x',x_n+f(x')\right) \\
&+ 2\left[\left(\partial^2_{x_j x_n} w\right)\left(x',x_n+f(x')\right)\right]\cdot\left[\partial_{x_j} f(x')\right] \\
&+ \left[\left(\partial^2_{x_n} w\right)\left(x',x_n+f(x')\right)\right]\cdot\left[\partial_{x_j} f(x')\right]^2 \\
&= \partial^2_{x_j}\widetilde{w}(x',x_n) + 2\left[\partial^2_{x_j x_n}\widetilde{w}(x',x_n)\right]\cdot\left[\partial_{x_j} f(x')\right] \\
&+ \left[\partial^2_{x_n}\widetilde{w}(x',x_n)\right]\cdot\left[\partial_{x_j} f(x')\right]^2 .
\end{aligned}
\tag{C.1.24}
$$

For $j = n$, it is immediate to see that

$$
\begin{aligned}
\partial_{x_n}(w(x',x_n+f(x'))) &= \partial_{x_n}\widetilde{w}(x',x_n), \\
\partial^2_{x_n}(w(x',x_n+f(x'))) &= \partial^2_{x_n}\widetilde{w}(x',x_n).
\end{aligned}
\tag{C.1.25}
$$

We conclude the proof by taking into account (C.1.22) and combining (C.1.24)-(C.1.25) into (C.1.23). □

The second step will be to make another change of variables in order to reduce the operator given by (C.1.20) in another of the form (C.1.11).

Lemma C.1.7. *Consider the change of variables Υ_2 from $\mathbb{R}^n_0$ onto $\mathbb{R}^n_{+1}$, defined by*

$$\Upsilon_2(x',x_n) = (x', e^{x_n}) =: (x', r).$$

Then the operator $\mathcal{L}_{0,0,\varphi,\varepsilon}$ in (x',r)-coordinates has the form:

$$\mathcal{L}_{0,0,\varphi,\varepsilon,\Upsilon_1,\Upsilon_2} = r^2\,\mathcal{L}_{\tau,\varepsilon,\Upsilon_2\circ\Upsilon_1} + \tau^{-1} J_{\Upsilon_2\circ\Upsilon_1}, \tag{C.1.26}$$

where

$$
\begin{aligned}
\mathcal{L}_{\tau,\varepsilon,\Upsilon_2\circ\Upsilon_1} = \left(1+|\nabla_{x'} f|^2\right)\tau^{-2}\partial_r^2 &- \frac{2}{r}\left(\alpha_{\Upsilon_2\circ\Upsilon_1} + \nabla_{x'} f\cdot\tau^{-1}\nabla_{x'}\right)\tau^{-1}\partial_r \\
&+ \frac{1}{r^2}\left(\alpha^2_{\Upsilon_2\circ\Upsilon_1} + \tau^{-2}\Delta_{x'}\right),
\end{aligned}
\tag{C.1.27}
$$

with

$$\alpha_{\Upsilon_2\circ\Upsilon_1} = 1 + \frac{\tau^{-1}}{\varepsilon}(\log r + f(x')) \tag{C.1.28}$$

and $J_{\Upsilon_2\circ\Upsilon_1}$ is the first-order operator defined by

$$J_{\Upsilon_2\circ\Upsilon_1} = \tau^{-1} r\,\Delta_{x'} f\,\partial_r - \frac{\tau^{-1}}{\varepsilon}. \tag{C.1.29}$$

Proof. We start by noting that Υ_2 only changes the x_n-variable and then we only have to compute the derivates ∂_{x_n} and $\partial^2_{x_n}$ in terms of ∂_r and ∂^2_r. Now consider a smooth function $\underline{w}$ on $\Upsilon_2(\mathbb{R}^n_0)$. Then, by the chain rule, we have

$$\begin{aligned}
\partial_{x_n}\left[\underline{w}(x',r)\right] &= \left[\partial_r \underline{w}(x',r)\right]\cdot\left[\partial_{x_n} r\right] = r\,\partial_r \underline{w}(x',r),\\
\partial^2_{x_n}\left[\underline{w}(x',r)\right] &= r\,\partial_r\left[r\partial_r \underline{w}(x',r)\right] = \left(r\,\partial_r + r^2\partial_r^2\right)\underline{w}(x',r).
\end{aligned}$$

We conclude the proof by combining these identities into (C.1.20). ☐

Proof of Proposition C.1.5. We start by giving the outline of the proof. The first step will be to apply Lemma C.1.2 to the operator $\mathcal{L}_{\tau,\varepsilon,\Upsilon_1\circ\Upsilon_2}$, see (C.1.27), in order to get the following estimate

$$\frac{\tau^{-1}}{\sqrt{\varepsilon}}\left\|v\right\|_{L^2(\Upsilon_2\circ\Upsilon_1(\Omega))} \leqslant C\left\|\mathcal{L}_{\tau,\varepsilon,\Upsilon_1\circ\Upsilon_2} v\right\|_{H^{-1}_{scl}(\mathbb{R}^n_{+1})},$$

for all $v\in C_0^\infty(\Upsilon_2\circ\Upsilon_1(\Omega))$. The second step will be to transfer a similar estimate as above, now to the operator $\mathcal{L}_{0,0,\varphi,\varepsilon,\Upsilon_1,\Upsilon_2}$, see (C.1.26). The third and last step will be to undo the change of variables $\Upsilon_2\circ\Upsilon_1$ to go from $\mathcal{L}_{0,0,\varphi,\varepsilon,\Upsilon_1,\Upsilon_2}$ to $\mathcal{L}_{0,0,\varphi,\varepsilon}$ and then get the statement of this proposition.

First step. Let Ω_2 be a smooth bounded open set slightly larger than Ω such that

$$\Omega\subsetneq\Omega_2,\quad E\subset\partial\Omega\cap\partial\Omega_2.$$

Now we set $U=\Upsilon_2\circ\Upsilon_1(\Omega)$ and $U_2=\Upsilon_2\circ\Upsilon_1(\Omega_2)$. Then it is clear that

$$U\subsetneq U_2,\quad \emptyset\neq\Upsilon_2\circ\Upsilon_1(E)\subset\partial U\cap\partial U_2\subset\partial\mathbb{R}^n_{+1}.$$

Moreover by Lemma C.1.7, the operator $\mathcal{L}_{\tau,\varepsilon,\Upsilon_2\circ\Upsilon_1}$ is a second-order semiclassical operator of the form (C.1.11) with the variable x' instead of θ and

$$F=G=\nabla_{x'}f,\quad a=\alpha_{\Upsilon_2\circ\Upsilon_1},\quad L_\theta=\Delta_{x'}.$$

It remains to verify the conditions (C.1.12)-(C.1.16). We start to verify (C.1.12). On Ω_2 we apply the Carleman estimate obtained by Dos Santos Ferreira et al. [17], see Proposition 2.3 therein, more precisely their estimate (2.12); to deduce that there exists $C_3>0$ such that the following estimate

$$\frac{\tau^{-1}}{\sqrt{\varepsilon}}\left\|w\right\|_{H^1_{scl}(\Omega_2)} \leqslant C_3\left\|\mathcal{L}_{0,0,\varphi,\varepsilon} w\right\|_{L^2(\Omega_2)}, \tag{C.1.30}$$

holds true for all $w\in C_0^\infty(\Omega_2)$. On the other hand, from (C.1.29) it is immediate to see that

$$\left\|J_{\Upsilon_2\circ\Upsilon_1} v\right\|_{L^2(U_2)} \leqslant C_4\left\|v\right\|_{H^1_{scl}(U_2)},\quad v\in C_0^\infty(U_2). \tag{C.1.31}$$

Now let $v \in C_0^\infty(U_2)$ be a fixed function. Then there exists $w \in C_0^\infty(\Omega_2)$ such that $v = w \circ \Upsilon^{-1} \circ \Upsilon^{-2}$. Hence, from (C.1.26) and (C.1.30)-(C.1.31), we obtain

$$\begin{aligned}
\|\mathcal{L}_{\varphi,\varepsilon,\Upsilon_1\circ\Upsilon_2} v\|_{L^2(U_2)} &= \left\|r^{-2}\mathcal{L}_{0,0,\varphi,\varepsilon,\Upsilon_1,\Upsilon_2} v - r^{-2}\tau^{-1} J_{\Upsilon_2\circ\Upsilon_1} v\right\|_{L^2(U_2)} \\
&\geqslant C_5 \|\mathcal{L}_{0,0,\varphi,\varepsilon,\Upsilon_1,\Upsilon_2} v\|_{L^2(U_2)} - C_5\tau^{-1} \|v\|_{H^1_{scl}(U_2)} \\
&\geqslant C_6 \|\mathcal{L}_{0,0,\varphi,\varepsilon} w\|_{L^2(\Omega_2)} - C_6\tau^{-1} \|w\|_{H^1_{scl}(\Omega_2)} \\
&\geqslant C_7 \frac{\tau^{-1}}{\sqrt{\varepsilon}} \|w\|_{H^1_{scl}(\Omega_2)} - C_6\tau^{-1} \|w\|_{H^1_{scl}(\Omega_2)} \\
&\geqslant C_8 \frac{\tau^{-1}}{\sqrt{\varepsilon}} \|w\|_{H^1_{scl}(\Omega_2)} \geqslant C_9 \frac{\tau^{-1}}{\sqrt{\varepsilon}} \|v\|_{H^1_{scl}(U_2)},
\end{aligned}$$

where in the last line we have taken ε small enough. In the above estimates, we have also used that

$$\|\mathcal{L}_{0,0,\varphi,\varepsilon,\Upsilon_1,\Upsilon_2} v\|_{L^2(U_2)} \simeq \|\mathcal{L}_{0,0,\varphi,\varepsilon} w\|_{L^2(\Omega_2)}, \quad \|v\|_{H^1_{scl}(U_2)} \simeq \|w\|_{H^1_{scl}(\Omega_2)}.$$

Hence, the operator $\mathcal{L}_{\varphi,\varepsilon,\Upsilon_1\circ\Upsilon_2}$ satisfies (C.1.12). We now turn to verify the remainder conditions. From the hypothesis, we have

$$||\nabla_{x'} f| - |K|| \leqslant |\nabla_{x'} f - K| \leqslant \delta.$$

Hence, the conditions (C.1.13) and (C.1.14) are satisfied. Since L_θ is the Laplacian operator $\Delta_{x'}$, the condition (C.1.15) is trivially satisfied. Finally, from (C.1.28), the condition (C.1.16) is also satisfied. Hence, applying Lemma (C.1.2), we get

$$\frac{\tau^{-1}}{\sqrt{\varepsilon}} \|v\|_{L^2(U)} \leqslant C \|\mathcal{L}_{\tau,\varepsilon,\Upsilon_1\circ\Upsilon_2} v\|_{H^{-1}_{scl}(\mathbb{R}^n_{+1})}, \quad v \in C_0^\infty(U). \tag{C.1.32}$$

Second step. Now we will obtain the above estimate with $\mathcal{L}_{0,0,\varphi,\varepsilon,\Upsilon_1,\Upsilon_2}$ instead of $\mathcal{L}_{\tau,\varepsilon,\Upsilon_1\circ\Upsilon_2}$. Let $v \in C_0^\infty(U)$ be a fixed function. Since $J_{\Upsilon_2\circ\Upsilon_1}$ is a first-order semiclassical operator, (C.1.26) and (C.1.32), we get

$$\begin{aligned}
\|\mathcal{L}_{\tau,\varepsilon,\Upsilon_1\circ\Upsilon_2} v\|_{H^{-1}_{scl}(\mathbb{R}^n_{+1})} &= \left\|r^2\, \mathcal{L}_{\tau,\varepsilon,\Upsilon_2\circ\Upsilon_1} v + \tau^{-1} J_{\Upsilon_2\circ\Upsilon_1} v\right\|_{H^{-1}_{scl}(\mathbb{R}^n_{+1})} \\
&\geqslant C_{10} \|\mathcal{L}_{\tau,\varepsilon,\Upsilon_1\circ\Upsilon_2} v\|_{H^{-1}_{scl}(\mathbb{R}^n_{+1})} - C_{10}\tau^{-1} \|v\|_{L^2(\mathbb{R}^n_{+1})} \\
&\geqslant C_{11} \frac{\tau^{-1}}{\sqrt{\varepsilon}} \|v\|_{L^2(U)} - C_{10}\tau^{-1} \|v\|_{L^2(U)} \\
&\geqslant C_{12} \frac{\tau^{-1}}{\sqrt{\varepsilon}} \|v\|_{L^2(U)},
\end{aligned}$$

where at the last line we have again taken ε small enough.

Third step. Now we undo the change of variables given by Υ_1 and Υ_2. For every $w \in C_0^\infty(\Omega)$ there exists $v \in C_0^\infty(U)$ such that $w = v \circ \Upsilon_2 \circ \Upsilon_1$. Then, from the above estimate we have

$$\|\mathcal{L}_{0,0,\varphi,\varepsilon}w\|_{H^{-1}_{scl}(\Xi_f)} \geqslant C_{13}\,\|\mathcal{L}_{\tau,\varepsilon,\Upsilon_1\circ\Upsilon_2}v\|_{H^{-1}_{scl}(\mathbb{R}^n_{+1})}$$
$$\geqslant C_{14}\frac{\tau^{-1}}{\sqrt{\varepsilon}}\,\|v\|_{L^2(U)} \geqslant C_{14}\frac{\tau^{-1}}{\sqrt{\varepsilon}}\,\|w\|_{L^2(\Omega)}\,.$$

This completes the proof. □

The next step will be to remove the smallness condition (C.1.18) from Proposition C.1.5 and also obtain the estimate (C.1.19) for the full operator $\mathcal{L}_{A,q,\varphi,\varepsilon}$.

Proposition C.1.8. *Consider the linear limiting Carleman weight $\varphi(x) = x_n$. Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be a C^∞ function such that*

$$E \subset \left\{x = (x', x_n)\,:\, x_n = f(x')\right\}$$

and $\Omega \subset \Xi_f$, where Ξ_f is defined by

$$\Xi_f := \left\{x = (x', x_n)\,:\, x_n \geqslant f(x')\right\}.$$

Then there exists a constant $C > 0$ (depending on f and Ω) such that for all $w \in C_0^\infty(\Omega)$, we have

$$\frac{\tau^{-1}}{\sqrt{\varepsilon}}\,\|w\|_{L^2(\Omega)} \leqslant C\,\|\mathcal{L}_{A,q,\varphi,\varepsilon}w\|_{H^{-1}_{scl}(\Xi_f)}\,. \tag{C.1.33}$$

Proof. This Proposition is analogous of Proposition 8.1 in [14] and so we follow the proof given there. We divide the proof into two steps. The first step will be to prove (C.1.33) for the operator $\mathcal{L}_{0,0,\varphi,\varepsilon}$. To do that, we define the smooth function $G(x', x_n) = x_n - f(x')$. Thus, for any ball $B_\delta(x_0)$ of radius $\delta > 0$ and centered at any fixed point $x_0 \in \mathbb{R}^n$, we have the following inequality

$$\left|\nabla f(x') - \nabla f(y\prime)\right| = |\nabla G(x) - \nabla G(y)| \leqslant C_5\,|x-y| \leqslant 2\delta C_5,\ \ x, y \in B_\delta(x_0).$$

Hence, taking $\delta > 0$ small enough, the function f satisfies the condition (C.1.18) on $B_\delta(x_0)$. Now since $\overline{\Omega}$ is a compact set in $\mathbb{R}^n$, we can take a finite open cover of Ω denoted by $\left\{B_{\delta_j}(x_j)\right\}_{j=1}^m$ and such that

$$|\nabla f - K_j| \leqslant C_j\delta_j, \quad \text{on } \Omega \cap B_{\delta_j}(x_j)$$

for some fixed $K_j \in \mathbb{R}^n$ with $j = 1, \ldots, m$. Now we apply Proposition C.1.5 on each $\Omega_j := \Omega \cap B_{\delta_j}(x_j)$ and taking into account that $U_j \subset \Xi_f$. Thus, there exist a positive constants τ_1 and C_j with $j = 1, \ldots, m$, such that the inequality

$$\frac{\tau^{-1}}{\sqrt{\varepsilon}}\,\|v\|_{L^2(\Omega_j)} \leqslant C_j\,\|\mathcal{L}_{0,0,\varphi,\varepsilon}v\|_{H^{-1}_{scl}(\Xi_f)}\,, \tag{C.1.34}$$

holds true for all $\tau \geqslant \tau_1$ and for all $v \in C_0^\infty(\Omega_j)$. Now applying the partition of the unity Theorem subordinate to $(U_j)_{j=1}^m$, there exists a family of smooth non-negative functions $(\gamma_j)_{j=1}^m$ such that

$$\sum_{j=1}^m \gamma_j = 1 \quad \text{and} \quad \operatorname{supp} \gamma_j \subset U_j.$$

Consider $w \in C_0^\infty(\Omega)$ and by the above conditions, it can be written as $w = \sum_{j=1}^m w\gamma_j$. Notice also that $w\gamma_j \in C_0^\infty(U_j)$ for all $j = 1, \dots, m$. Hence, applying (C.1.34) to each one of these previous functions, we obtain

$$\frac{\tau^{-1}}{\sqrt{\varepsilon}} \left\| w\gamma_j \right\|_{L^2(\Omega_j)} \leqslant C_j \left\| \mathcal{L}_{0,0,\varphi,\varepsilon}(w\gamma_j) \right\|_{H^{-1}_{scl}(\Xi_f)}, \quad j = 1, \dots, m. \tag{C.1.35}$$

For a moment we leave this inequality to remark two facts. The first fact is relative to an obvious inequality. Since $\operatorname{supp}(w\gamma_j) \subset U_j$, we have

$$\int_\Omega |w\gamma_j|^2 \, dx = \int_{\operatorname{supp}(w\gamma_j)} |w\gamma_j|^2 \, dx \leqslant \int_{U_j} |w\gamma_j|^2 \, dx,$$

which implies

$$\|w\|_{L^2(\Omega)} \leqslant \sum_{j=1}^m \|w\gamma_j\|_{L^2(\Omega)} \leqslant \sum_{j=1}^m \|w\gamma_j\|_{L^2(U_j)}. \tag{C.1.36}$$

The second fact is relative to the $H^{-1}_{scl}(\Xi_f)$-norm from the right-hand side of (C.1.34). From (C.1.22) and the chain rule, we have

$$\begin{aligned} \mathcal{L}_{0,0,\varphi,\varepsilon}(w\gamma_j) &= \gamma_j \, \mathcal{L}_{0,0,\varphi,\varepsilon} w \\ &\quad + \tau^{-1} \left[2\nabla\gamma_j \cdot \tau^{-1}\nabla - 2\left(1 + \frac{\tau^{-1}}{\varepsilon}\right) \partial_{x_n}\gamma_j + \tau^{-1}\Delta\gamma_j \right] w, \end{aligned}$$

which implies

$$\begin{aligned} \left\| \mathcal{L}_{0,0,\varphi,\varepsilon}(w\gamma_j) \right\|_{H^{-1}_{scl}(\Xi_f)} &\leqslant \left\| \mathcal{L}_{0,0,\varphi,\varepsilon} w \right\|_{H^{-1}_{scl}(\Xi_f)} + C_6 \, \tau^{-1} \|w\|_{L^2(\Xi_f)} \\ &\leqslant \left\| \mathcal{L}_{0,0,\varphi,\varepsilon} w \right\|_{H^{-1}_{scl}(\Xi_f)} + C_6 \, \tau^{-1} \|w\|_{L^2(\Omega)}, \end{aligned} \tag{C.1.37}$$

where in the last inequality we have used that $\operatorname{supp} w \subset \Omega$. Now we return to inequality (C.1.35), by summing both sides from $j = 1$ to $j = m$ and taking into account (C.1.36)-(C.1.37), we obtain

$$\frac{\tau^{-1}}{\sqrt{\varepsilon}} \|w\|_{L^2(\Omega)} \leqslant C_7 \left\| \mathcal{L}_{0,0,\varphi,\varepsilon} w \right\|_{H^{-1}_{scl}(\Xi_f)} + C_7 \, \tau^{-1} \|w\|_{L^2(\Omega)}, \quad w \in C_0^\infty(\Omega).$$

The second term of the right-hand side of the above inequality can be absorbed into the left-hand side by taking ε small enough. Hence, we have

$$\frac{\tau^{-1}}{\sqrt{\varepsilon}} \|w\|_{L^2(\Omega)} \leqslant C_8 \left\| \mathcal{L}_{0,0,\varphi,\varepsilon} w \right\|_{H^{-1}_{scl}(\Xi_f)}. \tag{C.1.38}$$

The second and last step will be to prove that the above inequality still holds for the full operator $\mathcal{L}_{A,q,\varphi,\varepsilon}$. Indeed, consider $w \in C_0^\infty(\Omega)$ and by a straightforward computation we obtain

$$\begin{aligned}&\mathcal{L}_{A,q,\varphi,\varepsilon} w = \mathcal{L}_{0,0,\varphi,\varepsilon} w \\ &+ \tau^{-1}\left[-2iA\cdot\tau^{-1}\nabla + 2iA\cdot e_n\left(1+\frac{\tau^{-1}}{\varepsilon}x_n\right) + \tau^{-1}\left(-i\nabla\cdot A + A^2 + q\right)\right] w.\end{aligned}$$

Combining this identity with (C.1.38), we get

$$\begin{aligned}\frac{\tau^{-1}}{\sqrt{\varepsilon}}\|w\|_{L^2(\Omega)} &\leqslant C_8 \|\mathcal{L}_{0,0,\varphi,\varepsilon} w\|_{H^{-1}_{scl}(\Xi_f)} \\ &\leqslant C_8 \|\mathcal{L}_{A,q,\varphi,\varepsilon} w\|_{H^{-1}_{scl}(\Xi_f)} + C_9\,\tau^{-1}\|w\|_{L^2(\Xi_f)} \\ &\leqslant C_8 \|\mathcal{L}_{A,q,\varphi,\varepsilon} w\|_{H^{-1}_{scl}(\Xi_f)} + C_9\,\tau^{-1}\|w\|_{L^2(\Omega)}.\end{aligned}$$

We end the proof by taking ε small enough to absorb the second term of the right-hand side of this inequality into the left-hand side. □

This proposition can be used to prove a Carleman estimate, which will then be the main tool to construct CGO solutions in Ω of the equation $\mathcal{L}_{A,q,\varphi} u = 0$ with the desired vanishing condition on $E \subset \partial\Omega$. More precisely, we have the following Carleman estimate.

Corollary C.1.9. *Let $\xi \in \mathbb{S}^n$ be given and consider the linear limiting Carleman weight $\varphi(x) = \xi\cdot x$. Suppose that Ω' is a smooth domain with $\Omega \subset \Omega'$ such that $\partial\Omega' \cap \partial\Omega = E$. Then there exist two positive constants C and τ_0 (depending on n, Ω and priori bounds on A and q) such that the following estimate*

$$\tau^{-1}\|w\|_{L^2(\Omega)} \leqslant C\|\mathcal{L}_{A,q,\varphi} w\|_{H^{-1}_{scl}(\Omega')}, \quad w \in C_0^\infty(\Omega)$$

holds true for all $\tau \geqslant \tau_0$.

Proof. Since $\overline{\Omega}$ is a compact set, there exists a finite family of open sets, $\{B_j\}_{j=1}^m$, and also a finite family of real valued-functions defined on $\mathbb{R}^{n-1}$, $\{f_j\}_{j=1}^m$, such that

$$\Omega \subset \overline{\Omega} \subset \bigcup_{j=1}^m B_j,$$

and

$$\partial\Omega \cap U_j = \left\{x \in \mathbb{R}^n \,:\, x_n = f_j(x')\right\}, \quad \Omega \cap U_j \subset \Xi_j, \quad j = 1,\dots,m,$$

where $\Xi_j = \{x \in \mathbb{R}^n : x_n \geqslant f_j(x')\}$ with $j = 1,\dots,m$. Now for each $j \in \{1,\dots,m\}$, Proposition C.1.8 ensures that there exist positive constants τ_1 and C_j with $j = 1,\dots,m$ such that the estimate

$$\frac{\tau^{-1}}{\sqrt{\varepsilon}}\|w\|_{L^2(\Omega\cap B_j)} \leqslant C_j\|\mathcal{L}_{A,q,\varphi,\varepsilon} w\|_{H^{-1}_{scl}(\Xi_j)}, \tag{C.1.39}$$

holds true for all $w \in C_0^\infty(\Omega \cap B_j)$ and for all $\tau \geqslant \tau_1$. We would like to have the norm $H^{-1}_{scl}(\Omega')$ in the above inequality instead of the norm $H^{-1}_{scl}(\Xi_j)$. The following claim say us that this is possible.

Claim. For each $j \in \{1, \dots, m\}$, the identity map from $H^{-1}_{scl}(\Omega')$ to $H^{-1}_{scl}(\Xi_j)$ is bounded on $C_0^\infty(\Omega \cap B_j)$.

Indeed, we proceed by using a cutoff argument. For a fixed $j \in \{1, \dots, m\}$, we consider the cutoff function:

$$\chi_j = \begin{cases} 1, & \text{in } \ \Omega \cap B_j, \\ 0, & \text{in } \ \Xi_j^c \cap \Omega'^c. \end{cases} \tag{C.1.40}$$

Consequently, if $v \in H_0^1(\Xi_j)$ then $\chi_j\, v \in H_0^1(\Omega')$. Moreover, the operator $T_{\chi_j} : H_0^1(\Xi_j) \to H_0^1(\Omega')$ defined by $v \mapsto \chi_j v$ is bounded, that is, there exists $C_5 > 0$ such that:

$$\left\| T_{\chi_j} v \right\|_{H_0^1(\Omega')} = \left\| \chi_j v \right\|_{H_0^1(\Omega')} \leqslant C_5 \left\| v \right\|_{H_0^1(\Xi_j)}, \quad v \in H_0^1(\Xi_j). \tag{C.1.41}$$

For a fix $w \in C_0^\infty(\Omega \cap B_j)$ and from the definition of the cutoff function χ_j, we obtain the following identity for all $\psi \in C_0^\infty(\Xi_j) \setminus \{0\}$:

$$\begin{aligned} \langle w, \psi \rangle_{\Xi_j} &= \int_{\Xi_j} w\, \psi dx = \int_{\Xi_j \cap \operatorname{supp} w} w\, \psi dx \\ &= \int_{\Xi_j \cap \operatorname{supp} w} w\, \chi_j \psi dx = \int_{\Omega'} w\, \chi_j \psi dx = \langle w, \chi_j \psi \rangle_{\Omega'} . \end{aligned}$$

By using the following facts: if $\psi \in C_0^\infty(\Xi_j) \setminus \{0\}$ then $\chi_j \psi \in C_0^\infty(\Omega') \setminus \{0\}$, the definition of the norm in $H^{-1}_{scl}(\Omega')$ (see C.1.10), the above identity and (C.1.41); we get

$$\begin{aligned} \frac{\left| \langle w, \psi \rangle_{\Xi_j} \right|}{\|\psi\|_{H^1_{scl}(\Xi_j)}} &= \frac{\left| \langle w, \chi_j \psi \rangle_{\Omega'} \right|}{\|\psi\|_{H^1_{scl}(\Xi_j)}} \leqslant C_5 \frac{\left| \langle w, \chi_j \psi \rangle_{\Omega'} \right|}{\|\chi_j \psi\|_{H^1_{scl}(\Omega')}} \\ &\leqslant C_5 \sup_{\phi \in C_0^\infty(\Omega') \setminus \{0\}} \frac{\left| \langle u, \phi \rangle_{\Omega'} \right|}{\|\phi\|_{H^1_{scl}(\Omega')}} = C_5 \left\| w \right\|_{H^{-1}(\Omega')} . \end{aligned}$$

Since the above inequality holds for all $\psi \in C_0^\infty(\Xi_j) \setminus \{0\}$, and now from the definition of the norm in $H^{-1}_{scl}(\Xi_j)$, we get

$$\left\| w \right\|_{H^{-1}_{scl}(\Xi_j)} \leqslant C_5 \left\| w \right\|_{H^{-1}_{scl}(\Omega')}, \quad w \in C_0^\infty(\Omega \cap B_j),$$

which proves the claim.

Combining this claim with (C.1.39), we get the following estimate for each $j \in \{1, \dots, m\}$

$$\frac{\tau^{-1}}{\sqrt{\varepsilon}} \left\| w \right\|_{L^2(\Omega \cap B_j)} \leqslant C_6 \left\| \mathcal{L}_{A,q,\varphi,\varepsilon} w \right\|_{H^{-1}_{scl}(\Omega')}, \quad w \in C_0^\infty(\Omega \cap B_j).$$

Now we can proceed by a partition of the unit argument subordinate to the family $\{\Omega \cap B_j\}_{j=1}^m$, as was done in the proof of Proposition C.1.8, to deduce

$$\frac{\tau^{-1}}{\sqrt{\varepsilon}} \|w\|_{L^2(\Omega)} \leqslant C_7 \|\mathcal{L}_{A,q,\varphi,\varepsilon} w\|_{H^{-1}_{scl}(\Omega')}, \quad w \in C_0^\infty(\Omega). \tag{C.1.42}$$

The last step will be to remove the exponential terms $e^{\pm\varphi^2/2\varepsilon}$ of the operator $\mathcal{L}_{A,q,\varphi,\varepsilon}$ to have (C.1.42) with $\mathcal{L}_{A,q,\varphi}$. This can be done by combining two facts: there exists $C_8 > 0$ (only depending on Ω') such that $1 \leqslant e^{\varphi^2/2\varepsilon} \leqslant e^{C_8/\varepsilon}$ in Ω' and also, that if $w \in C_0^\infty(\Omega)$ then $e^{\varphi^2/2\varepsilon} w \in C_0^\infty(\Omega)$. Thus, from (C.1.42) applied to $e^{\varphi^2/2\varepsilon} w$ with $w \in C_0^\infty(\Omega)$, we obtain

$$\begin{aligned}\frac{\tau^{-1}}{\sqrt{\varepsilon}} \|w\|_{L^2(\Omega)} &\leqslant \frac{\tau^{-1}}{\sqrt{\varepsilon}} \left\|e^{\varphi^2/2\varepsilon} w\right\|_{L^2(\Omega)} \leqslant C_7 \left\|e^{\varphi^2/2\varepsilon} \mathcal{L}_{A,q,\varphi,\varepsilon} w\right\|_{H^{-1}_{scl}(\Omega')} \\ &\leqslant C_9\, e^{C_8/\varepsilon} \|\mathcal{L}_{A,q,\varphi} w\|_{H^{-1}_{scl}(\Omega')}.\end{aligned}$$

This completes the proof. □

We use Corollary C.1.9 to construct solutions $u \in H^1(\Omega)$ (satisfying the vanishing condition on $E \subset \partial\Omega$) of the inhomogeneous adjoint equation $\mathcal{L}^*_{A,q,\varphi} u = v$ in Ω with $v \in L^2(\Omega)$. More precisely, we have:

Proposition C.1.10. *For every $v \in L^2(\Omega)$, there exists $u \in H^1(\Omega)$ satisfying*

$$\begin{cases} {\mathcal{L}^*}_{A,q,\varphi}\, u &= v, \ \text{in} \ \ \Omega, \\ u|_E &= 0. \end{cases} \tag{C.1.43}$$

Moreover, there exist two positive constants C and τ_0 such that

$$\|u(\cdot,\tau)\|_{H^1_{scl}(\Omega)} \leqslant C\,\tau\, \|v\|_{L^2(\Omega)}, \tag{C.1.44}$$

for all $\tau \geqslant \tau_0$.

Proof. The proof is standard and it follows by using a Hahn-Banach argument. It is easy to see that $\mathcal{L}_{A,q,\varphi}\left(C_0^\infty(\Omega)\right) \subset H^{-1}_{scl}(\Omega')$. Now we claim that the operator $T : \mathcal{L}_{A,q,\varphi}\left(C_0^\infty(\Omega)\right) \to \mathbb{R}$ given by

$$T(\mathcal{L}_{A,q,\varphi}\, w) = \langle w, v\rangle_\Omega\,,$$

is a well defined and bounded linear functional. In fact, by Corollary C.1.9 and Cauchy-Schwarz inequality, we have

$$\begin{aligned}|T(\mathcal{L}_{A,q,\varphi}\, w)| &= |\langle w, v\rangle_\Omega| \leqslant \tau\, \|v\|_{L^2(\Omega)}\, \tau^{-1} \|w\|_{L^2(\Omega)} \\ &\leqslant C\tau \|v\|_{L^2(\Omega} \|\mathcal{L}_{A,q,\varphi} w\|_{H^{-1}_{scl}(\Omega')},\end{aligned}$$

which proof the claim. Moreover, we deduce the following estimate for the norm of T

$$\|T\|_{\mathcal{L}_{A,q,\varphi}\left(C_0^\infty(\Omega)\right)\to\mathbb{R}} \leqslant C\tau \|v\|_{L^2(\Omega)}. \tag{C.1.45}$$

Then, the Hahn-Banach Theorem ensures that there exists an extension of T to the whole space $H^{-1}_{scl}(\Omega')$, $\widetilde{T} : H^{-1}_{scl}(\Omega') \to \mathbb{R}$, preserving the norm of T, that is

$$\left\|\widetilde{T}\right\|_{H^{-1}_{scl}(\Omega')\to\mathbb{R}} = \|T\|_{\mathcal{L}_{A,q,\varphi}\left(C_0^\infty(\Omega)\right)\to\mathbb{R}} . \tag{C.1.46}$$

Now by Riesz Representation theorem applied to $\widetilde{T}$, there exists $u \in H^1_0(\Omega') \cap H^1_{scl}(\Omega')$ such that

$$\widetilde{T}f = \langle f, u\rangle_{\Omega'} , \ f \in H^{-1}_{scl}(\Omega') \ \text{ and } \ \left\|\widetilde{T}\right\|_{H^{-1}_{scl}(\Omega')\to\mathbb{R}} = \|u\|_{H^1_{scl}(\Omega')} . \tag{C.1.47}$$

Notice that such u also belongs to $H^1(\Omega)$ and since $E \subset \partial\Omega'$ it follows that $u|_E = 0$. It remains to check that also satisfies $\mathcal{L}^*_{A,q,\varphi}u = v$ in Ω. This can be deduced by considering $w \in C_0^\infty(\Omega)$ and the following estimate

$$\langle w, v\rangle_\Omega = T(\mathcal{L}_{A,q,\varphi}w) = \widetilde{T}(\mathcal{L}_{A,q,\varphi}w) = \langle \mathcal{L}_{A,q,\varphi}w, u\rangle_{\Omega'} = \left\langle w, \mathcal{L}^*_{A,q,\varphi}u\right\rangle_\Omega .$$

Finally, we deduce (C.1.44) by combining (C.1.45)-(C.1.47). The proof is completed. □

Finally, we are in the position to prove the main result of this section, that is the construction of solutions $u \in H^1(\Omega)$ of the equation $\mathcal{L}_{A,q}u = 0$ (in Ω) vanishing on $E \subset \partial\Omega$.

C.1.1 Proof of Theorem C.1.1

The proof of this theorem is similar to the proof of Proposition 9.2 in [14]. We start with the following identity. Let ρ be a smooth complex valued function. Then a straightforward computation give us

$$e^{-\rho}\mathcal{L}_{A,q}(e^\rho v) = \mathcal{L}_{A,q}v + \Big(|D\rho|^2 + D^2\rho + 2A\cdot D\rho\Big)\, v + 2D\rho\cdot Dv. \tag{C.1.48}$$

Now we try a solution of the equation $\mathcal{L}_{A,q}\, u = 0$ of the form

$$u = e^{\tau(\xi\cdot x + i\zeta\cdot x)}(a + r) - e^{\tau l}b, \tag{C.1.49}$$

and the remainder of this proof will be to provide necessary conditions for the functions a, r, l, b such that u defined in (C.1.49) satisfied the equation

$$\begin{cases} \mathcal{L}_{A,q}\, u = 0 & \text{in } \Omega \\ u|_E = 0. \end{cases}$$

Applying (C.1.48) with $\rho(x) = \tau(\xi\cdot x + i\zeta\cdot x)$ and $v = a + r$, we obtain

$$\begin{aligned} &\tau^{-2}e^{-\tau(\xi\cdot x+i\zeta\cdot x)}\mathcal{L}_{A,q}\left[e^{\tau(\xi\cdot x+i\zeta\cdot x)}(a+r)\right] = \tau^{-2}\mathcal{L}_{A,q}\, a \\ &- 2\tau^{-1}\left[(\xi + i\zeta)\cdot\nabla a + i(\xi+i\zeta)\cdot A\, a\right] + \tau^{-2}e^{-\tau(\xi\cdot x+i\zeta\cdot x)}\mathcal{L}_{A,q}\left(e^{\tau(\xi\cdot x+i\zeta\cdot x)}r\right) \end{aligned}$$

Now using (C.1.48) with $\rho(x) = \tau l(x)$ and $v = b$, give us

$$\begin{aligned}
&\tau^{-2} e^{-\tau(\xi\cdot x + i\zeta\cdot x)} \mathcal{L}_{A,q}\left(e^{\tau l} b\right) = \tau^{-2} e^{-\tau(\xi\cdot x + i\zeta\cdot x)} e^{\tau l} e^{-\tau l} \mathcal{L}_{A,q}\left(e^{\tau l} b\right)\\
&= e^{-\tau(\xi\cdot x + i\zeta\cdot x)} e^{\tau l}\left[\tau^{-2}\mathcal{L}_{A,q} b + |Dl|^2 b + \tau^{-1}\left(2Dl\cdot Db + (2Dl\cdot A + D^2 l) b\right)\right].
\end{aligned}$$

With these computations at hand, we see that u defined by (C.1.49) satisfies $\mathcal{L}_{A,q}u = 0$ in Ω iff

$$\begin{aligned}
0 &= \tau^{-2} e^{-\tau(\xi\cdot x + i\zeta\cdot x)} \mathcal{L}_{A,q} u\\
&= \tau^{-2} e^{-\tau(\xi\cdot x + i\zeta\cdot x)} \left[\mathcal{L}_{A,q}\left(e^{\tau(\xi\cdot x + i\zeta\cdot x)}(a + r)\right) - \mathcal{L}_{A,q}(e^{\tau l} b)\right]\\
&= \tau^{-2}\mathcal{L}_{A,q}\, a - 2\tau^{-1}\left[(\xi + i\zeta)\cdot\nabla a + i(\xi + i\zeta)\cdot A\, a\right]\\
&\quad + \tau^{-2} e^{-\tau(\xi\cdot x + i\zeta\cdot x)} \mathcal{L}_{A,q}\left(e^{\tau(\xi\cdot x + i\zeta\cdot x)} r\right)\\
&\quad - e^{-\tau(\xi\cdot x + i\zeta\cdot x)} e^{\tau l}\left[\tau^{-2}\mathcal{L}_{A,q} b + |Dl|^2 b + \tau^{-1}(2Dl\cdot Db + (2Dl\cdot A + D^2 l) b)\right].
\end{aligned} \tag{C.1.50}$$

This identity lead us to consider the necessary conditions for the functions a, r, l and b (similarly to [14]) as follows.

Equation for a. We imposed a satisfying

$$(\xi + i\zeta)\cdot\nabla a + i(\xi + i\zeta)\cdot A\, a = 0, \quad \text{in } \Omega.$$

By making the ansatz $a = e^{\Phi}$, the function Φ have to satisfy the following equation

$$(\xi + i\zeta)\cdot\nabla\Phi + i(\xi + i\zeta)\cdot A = 0, \quad \text{in } \Omega. \tag{C.1.51}$$

We solve this equation by applying Proposition B.3.1. To do that, first we make a compactly supported extension of the magnetic field A on the whole space $\mathbb{R}^n$, preserving the smoothness. Such extension, denoted by $\widetilde{A} \in C_c^{2,\gamma}(\mathbb{R}^n)$, satisfies

$$\left\|\widetilde{A}\right\|_{C_c^{2,\gamma}(\mathbb{R}^n)} \leqslant C_{10}\,\|A\|_{C^{2,\gamma}(\overline{\Omega})}.$$

Then, by Proposition B.3.1 applied with $m = 2$ and $F = -i(\xi + i\zeta)\cdot\widetilde{A}$, there exists a function $\widetilde{\Phi}$ satisfying in $\mathbb{R}^n$ the following equation

$$(\xi + i\zeta)\cdot\nabla\widetilde{\Phi} + i(\xi + i\zeta)\cdot\widetilde{A} = 0.$$

Furthermore, there exists a positive constant $\underline{\gamma}$ with $\underline{\gamma} < \gamma$, such that

$$\left\|\widetilde{\Phi}\right\|_{C^{3,\underline{\gamma}}(\mathbb{R}^n)} \leqslant C_{11}\left\|\widetilde{A}\right\|_{C^{2,\gamma}(\mathbb{R}^n)} \leqslant C_{12}\,\|A\|_{C^{2,\gamma}(\overline{\Omega})}.$$

In particular, the restriction $\Phi := \widetilde{\Phi}|_{\Omega} \in C^{3,\underline{\gamma}}(\Omega)$ is a solution of equation (C.1.51). Thus, the estimate (C.1.3) follows from the above estimate. The estimate (C.1.2) follows from

Lemma B.1.1. Since $a = e^{\Phi}$, we deduce that $a \in C^{3,\underline{\gamma}}(\Omega)$. Notice that a is independent of τ.

From (C.1.50), the function defined by $R = e^{i\tau\zeta\cdot x}r$ has to satisfy

$$\begin{aligned}\mathcal{L}^{*}_{A,q,\varphi}R &:= \tau^{-2}e^{-\tau\xi\cdot x}\mathcal{L}_{A,q}\left(e^{\tau\xi\cdot x}R\right) = -\tau^{-2}e^{i\tau\zeta\cdot x}\mathcal{L}_{A,q}\,a \\ &+ e^{-\tau\xi\cdot x}e^{\tau l}\left[\tau^{-2}\mathcal{L}_{A,q}b + |Dl|^2\,b + \tau^{-1}\left(2Dl\cdot Db + (2Dl\cdot A +\ D^2 l)b\right)\right].\end{aligned} \tag{C.1.52}$$

This equation for R can be solved by using Proposition C.1.10. In particular, $R|_E = 0$ and so $r|_E = 0$ (these facts will be verified later on). From these facts and since u defined by (C.1.49) has to satisfy the vanishing condition on E, we must impose the following boundary condition

$$(e^{\tau(\xi\cdot x + i\zeta\cdot x)}a)|_E = (e^{\tau l}b)|_E.$$

This allows us to consider the functions l and b satisfying the boundary conditions: $l(x)|_E = (\xi\cdot x + i\zeta\cdot x)|_E$ and $b|_E = a|_E$. Moreover, in order to have a decay on τ of R, we have to ensure the decay on τ of the left-hand side of (C.1.52). To achieve that, we assumed that the terms $|Dl|^2$ and $Dl\cdot Db + Dl\cdot A\,b$ are small in a suitable sense. More precisely:

Equation for l. We consider the function l being a solution of the following equation

$$\begin{cases} |Dl|^2 = \mathcal{O}\left(dist(x,E)^{\infty}\right), \\ l|_E = (\xi\cdot x + i\zeta\cdot x)|_E, \\ (\partial_\nu l)|_E = -\nu\cdot(\xi + i\zeta)|_E. \end{cases} \tag{C.1.53}$$

The first condition of this equation means that for every $p \in \mathbb{N}$ there exist two positive constants $\varepsilon = \varepsilon(p)$ and $C = C(p,\Omega)$ such that

$$|Dl\cdot Dl| = |\nabla l\cdot\nabla l| \leqslant Cs^p, \quad z < \varepsilon. \tag{C.1.54}$$

Observe that $l(x) = (\xi + i\zeta)\cdot x$ satisfies only the first two conditions of (C.1.53). The reason for which we consider the third condition is to avoid this duplicate solution. This equation was solved by Chung, see Proposition 7.2 in [13] and also Proposition 9.2 in [14]. For the convenience of the reader, we shall give the proof of the existence of such a function l satisfying (C.1.53). We start by picking coordinates (t,z) in a neighborhood of E, where t is the coordinate over E and s is perpendicular to E and stands for $dist(x,E)$. As a heuristic first approach, we consider a function $\widetilde{l}$ of the following formal series form

$$\widetilde{l}(t,s) = \sum_{j=0}^{\infty} a_j(t)s^j,$$

where the smooth functions a_j will be determined by imposing that $\widetilde{l}$ satisfies the following equation

$$\begin{cases} \nabla\widetilde{l}\cdot\nabla\widetilde{l}=0, \\ \widetilde{l}|_E=(\xi\cdot x+i\zeta\cdot x)|_E, \\ (\partial_\nu\widetilde{l})|_E=-\nu\cdot(\xi+i\zeta)|_E. \end{cases} \tag{C.1.55}$$

From the boundary conditions, it is immediate to deduce that

$$a_0(t)=(\xi\cdot x+i\zeta\cdot x)|_E, \quad a_1(t)=-\nu\cdot(\xi+i\zeta)|_E. \tag{C.1.56}$$

The task now is to determine the functions a_j for $j\geqslant 2$. In (t,s)-coordinates, the gradient of $\widetilde{l}$ has the form

$$\nabla\widetilde{l}=\left(\sum_{j=0}^{\infty}\nabla_t a_j(t)s^j, \sum_{j=0}^{\infty}a_j(t)js^j\right).$$

Then

$$0=\nabla\widetilde{l}\cdot\nabla\widetilde{l}=\sum_{m=0}^{\infty}\left(\sum_{j+k=m}\nabla_t a_j\cdot\nabla_t a_k+(j+1)(k+1)a_{j+1}a_{k+1}\right)s^m,$$

which implies

$$\sum_{j+k=m}\nabla_t a_j\cdot\nabla_t a_k+(j+1)(k+1)a_{j+1}a_{k+1}=0, \quad m\in\mathbb{N}. \tag{C.1.57}$$

For a fixed $m\in\mathbb{N}$, this recursive formula can be used to determine the unknown function a_{m+1} from the knowledge of the previously known functions a_j with $j\leqslant m$. From the recursive formula, we have

$$(m+1)a_1a_{m+1}=-\sum_{j+k=m}\nabla_t a_j\cdot\nabla_t a_k-\sum_{\substack{j+k=m\\ j\neq 0}}(j+1)(k+1)a_{j+1}a_{k+1}.$$

Hence, to determine a_{m+1} it only remains to verify that $a_1\neq 0$ on E. To do that, notice that on E we have $\nu\cdot\xi>0$, the set E is also a compact subset of the boundary and finally from (C.1.56); we deduce that $|a_1|>\varepsilon_0>0$. Thus, from (C.1.56) and the above recursive formula, we are able to know a_j for all $j\in\mathbb{N}$.

Claim. For any $p\in\mathbb{N}$ there exists $C>0$ (depending on all $\|a_j\|_{C(\Omega)}$ with $j=1,\dots,p$) such that the function l_p defined by

$$l_p(t,s)=\sum_{j=0}^{p}a_j(t)s^j \tag{C.1.58}$$

satisfies

$$|\nabla l_p\cdot\nabla l_p|\leqslant Cs^p.$$

Indeed, from the recursive formula (C.1.57) and by a straightforward computation, we obtain

$$\nabla l_p \cdot \nabla l_p = \sum_{m=0}^{p-1} \left(\sum_{j+k=m} \nabla_t a_j \cdot \nabla_t a_k + (j+1)(k+1) a_{j+1} a_{k+1} \right) s^m + \mathcal{O}(s^p) = \mathcal{O}(s^p).$$

This proves the claim. Moreover, notice that $l_p(t,0) = a_0(t) = (\xi \cdot x + i\zeta \cdot x)|_E$ and also $\partial_s l_p(t,0) = a_1(t) = -\nu \cdot (\xi + i\zeta)|_E$. Hence l_p defined in (C.1.58) satisfies (C.1.53). Finally, we have

$$\begin{aligned} l_p(t,s) &= a_0(t) + a_1(t)s + \sum_{j=2}^{p} a_j(t) s^j \\ &= a_0(t) + s \left[a_1(t) + \sum_{j=1}^{p-1} a_{j+1}(t) s^j \right] \\ &= (\xi \cdot x + i\zeta \cdot x)|_E + z \left[-\nu \cdot (\xi + i\zeta)|_E + \sum_{j=1}^{p-1} a_{j+1}(t) s^j \right]. \end{aligned}$$

Hence, we get

$$\Re l = \xi \cdot x|_E - z \left[\nu \cdot \xi|_E - \sum_{j=1}^{p-1} \Re\, a_{j+1}(t) s^j \right]$$

and then since $\nu \cdot \xi|_E > \varepsilon_0 > 0$ and taking s in a small enough neighborhood of zero, we conclude that $\Re\, l_p = \xi \cdot x - k(x)$ with $k(x) \simeq dist(x,E)$ and so (C.1.4) is also proved. Notice that l is independent of τ.

Equation for b. Now we will prove the existence of a smooth function b satisfying

$$\begin{cases} 2Dl \cdot Db + (2Dl \cdot A + D^2 l) b = \mathcal{O}\left(dist(x,E)^2\right) & \text{in } \Omega, \\ b|_E = a|_E. \end{cases} \tag{C.1.59}$$

We try a solution of the form

$$b(t,s) = b_0(t) + b_1(t)s + b_2(t)s^2, \tag{C.1.60}$$

where b_j are functions which will be determined later on, $j = 0,1,2$. Notice that from the boundary condition we know $b_0(t) = a|_E$. Since $a \in C^{3,\underline{\gamma}}(\Omega)$ it follows that $b_0 \in C^{3,\underline{\gamma}}$. Here $\underline{\gamma} > 0$ is given by the analysis of the existence of the function a. It only remains to determine b_1 and b_2. At this point, there is a slight difference with the construction of l because the magnetic potential A has only integer derivative until the second order, then its Taylor series is not well-defined. For this reason, we will consider its residual approximation until the second derivative. Thus, for convenience and for future computations, we write A in the (t,s)-coordinates as follows

$$A(t,s) = \left(A_0'(t) + A_1'(t)s + R_A'(t,s),\ A_0^n(t) + A_1^n(t)s + R_A^n(t,s) \right), \tag{C.1.61}$$

where A'_j and R'_A are vector-valued functions in $\mathbb{R}^{n-1}$, A^n_j and R^n_A are real-valued functions; $j = 0, 1$. Since $A \in C^{2,\gamma}(\overline{\Omega})$ (in particular belongs to $C^2(\overline{\Omega})$) we deduce that

$$(A'_0, A^n_0) \in C^2, \quad (A'_1, A^n_1) \in C^1, \quad (R'_A, R^n_A) \in C^0. \tag{C.1.62}$$

Moreover, we have

$$\left\| \left(R'_A(t,s); R^n_A(t,s) \right) \right\| \leqslant C s^2. \tag{C.1.63}$$

The above constant $C > 0$ only depends on Ω and $\|A\|_{C^2(\Omega)}$. The following identity is immediate

$$2Dl \cdot Db + (2Dl \cdot A + \, D^2 l) b = - \left[2 \nabla l \cdot \nabla b + (2i \nabla l \cdot A + \, \Delta l) b \right].$$

For a fixed $p \in \mathbb{N}$ and from (C.1.58), we consider l of the following form

$$l(t,s) = l_p(t,s) = \sum_{j=0}^{p} a_j(t) s^j.$$

Then, substituting (C.1.60) and (C.1.61) into $2\nabla l \cdot \nabla b + (2i\nabla l \cdot A + \, \Delta l)b$, we get

$$2\nabla l \cdot \nabla b + (2i \nabla l \cdot A + \, \Delta l) b = d_0(t,s) + d_1(t,s) s + \mathcal{O}(s^2), \tag{C.1.64}$$

where we have used (C.1.63) to deduce the term $\mathcal{O}(s^2)$. The function d_0 and d_1 are defined by

$$d_0(t,z) = 2(\nabla_t a_0 \cdot \nabla_t b_0 + a_1 b_1) + \left[2i(\nabla_t a_0 \cdot A'_0 + a_1 A^n_0) + \Delta_t a_0 + 2a_2 \right] b_0 \tag{C.1.65}$$

and

$$\begin{aligned} d_1(t,z) = {} & 2(\nabla_t a_0 \cdot \nabla_t b_1 + \nabla_t a_1 \cdot \nabla_t b_0 + 2a_1 b_2 + 2a_2 b_1) \\ & + 2i \left[\nabla_t a_0 \cdot A'_0 + a_1 A^n_0 + \Delta_t a_0 + 2a_2 \right] b_1 \\ & + \left[2i(\nabla_t a_0 \cdot A'_1 + \nabla_t a_1 \cdot A'_0 + a_1 A^n_1 + 2a_2 A^n_0) + \Delta_t a_1 \right] b_0. \end{aligned} \tag{C.1.66}$$

Since $\xi \cdot \nu \neq 0$ on E it follows that $a_1 = -\nu \cdot (\xi + i\zeta) \neq 0$. Hence, by imposing $d_0 = 0$, we can divide by a_1 in (C.1.65) and then we shall know b_1. From (C.1.65) we deduce that $b_1 \in C^{2,\gamma}$. Once known b_1, by imposing $d_1 = 0$ and dividing again by a_1 in (C.1.66), we shall know b_2. Since b_2 involves the term $\nabla_t b_1$, we deduce that $b_2 \in C^{1,\gamma}$. Hence, the function b by (C.1.60) satisfies (C.1.59) and belongs to $C^{1,\gamma}(\Omega)$.

Equation for r. Once proved the existence of the function l and b and setting

$$\begin{aligned} w = -\tau^{-2} e^{i\tau\zeta\cdot x} \mathcal{L}_{A,q}\, a - e^{-\tau\xi\cdot x} e^{\tau l} & \Big[\tau^{-2} \mathcal{L}_{A,q} b + |Dl|^2 \, b \\ & + \tau^{-1} \left(2Dl \cdot Db + (2Dl \cdot A + \, D^2 l) b \right) \Big], \end{aligned} \tag{C.1.67}$$

the equation (C.1.52) becomes $\mathcal{L}^*_{A,q,\varphi} R = w$. Now by Proposition C.1.10, there exists $R(\cdot,\tau) \in H^1(\Omega)$ such that $R(\cdot,\tau)|_E = 0$ and

$$\|R(\cdot,\tau)\|_{H^1_{scl}(\Omega)} \leqslant C_{11}\, \tau \, \|w\|_{L^2(\Omega)}. \tag{C.1.68}$$

Now we claim that $\|w\|_{L^2(\Omega)} = \mathcal{O}(\tau^{-2})$. Indeed, we divide the analysis of $\|w\|_{L^2(\Omega)}$ into two cases.
First case. When $dist(x, E) \leqslant \tau^{-1/2}$. Here we consider (C.1.54) with $p = 4$ and so

$$|\nabla l \cdot \nabla l| \leqslant Cs^4 \leqslant C\tau^{-2}.$$

Moreover

$$\left|2Dl \cdot Db + (2Dl \cdot A + D^2 l)b\right| \leqslant Cs^2 \leqslant C\tau^{-1}.$$

Thus, since $\left|e^{-\tau\xi\cdot x}e^{\tau l}\right| = e^{-\tau k} \leqslant 1$ and from (C.1.67), (C.1.53) and (C.1.59), we get $|w(x)| \leqslant C_{12}\tau^{-2}$.
Second case. When $dist(x, E) > \tau^{-1/2}$. Now consider (C.1.54) with $p = 2$ and then

$$|\nabla l \cdot \nabla l| \leqslant C_{13}s^2.$$

Analogously to the previous case we also have

$$\left|2Dl \cdot Db + (2Dl \cdot A + D^2 l)b\right| \leqslant C_{14}s^2.$$

Hence, since $k(x) \simeq dist(x, E) \simeq s$ and $s > \tau^{-1/2}$, we have

$$\begin{aligned} &\left|e^{-\tau\xi\cdot x}e^{\tau l}\left[\tau^{-2}\mathcal{L}_{A,q}b + |Dl|^2\, b + \tau^{-1}\left(2Dl \cdot Db + (2Dl \cdot A + D^2 l)b\right)\right]\right| \\ &\qquad \leqslant C_{15}e^{-\tau\kappa}(\tau^{-2} + s^2 + \tau^{-1}s^2) \leqslant C_{16}\tau^{-2}s^{-2}(\tau^{-2} + s^2 + \tau^{-1}s^2) \leqslant C_{17}\tau^{-2}. \end{aligned}$$

We also deduce easily the following estimate

$$\left|\tau^{-2}e^{i\tau\zeta\cdot x}\mathcal{L}_{A,q}\, a\right| \leqslant C_{18}\tau^{-2}.$$

From (C.1.67) and combining the two above inequalities, we deduce that $|w(x)| \leqslant C_{18}\tau^{-2}$. This completes the claim.

Hence, from (C.1.68) we get $\|R(\cdot, \tau)\|_{H^1_{scl}(\Omega)} \leqslant C_{12}\,\tau^{-1}$. Thus, by an immediate computation and by the Semiclassical Trace Theorem, the function $r \in H^1(\Omega)$ defined by $r = e^{-i\tau\xi\cdot x}R$ satisfies (C.1.5). So the proof of Theorem C.1.1 is completed.

Bibliography

[1] G. Alessandrini, *Stable determination of conductivity by boundary measurements*, Appl. Analysis **27** (1988), 153–172.

[2] K. Astala, T. Iwaniec, G. Martin, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton University Press, 2009.

[3] R. Brown, *Global uniqueness in the impedance imaging problem for less regular conductivities*, SIAM J. Math. Anal. **27**(1996), 1049-1056.

[4] R. Brown, *Recovering the conductivity at the boundary from the Dirichlet to Neumann map: a pointwise result*, J. Inverse Ill-Posed Probl. **9**(2001), 567–574.

[5] A. Bukhgeim and G. Uhlmann, *Recovering a potential from partial Cauchy data*, Comm. Partial Differential Equations **27** (2002), 653–668.

[6] A. P. Calderón, *On an inverse boundary value problem*, Seminar on Numerical Analysis and its Applications to Continuum Physics, Rio de Janeiro, Sociedade Brasileira de Matematica, (1980), 65–73.

[7] P. Caro, *On an inverse problem in electromagnetism with local data: stability and uniqueness*, Inverse Probl. Imaging **5** (2011), 297–322.

[8] P. Caro and V. Pohjola, *Stability estimates for an inverse problem for the magnetic Schrödinger operator*, International Math. Research Notices **21** (2015), 11083–11116.

[9] P. Caro, A. García and J. M. Reyes, *Stability of the Calderón problem for less regular conductivities*, *J. Differential Equations*, **254** (2013), 469–492.

[10] P. Caro, D. Dos Santos Ferreira and A. Ruiz, *Stability estimates for the Radon transform with restricted data and applications*, Adv. in Mathematics **267** (2012), 523–564.

[11] P. Caro, D. Dos Santos Ferreira and A. Ruiz, *Stability estimates for the Calderon problem with partial data*, preprint arXiv:1405.1217, 30 pages, 2014.

[12] P. Caro and K. Rogers, *Global uniqueness for the Calderón problem with Lipschitz conductivities*, Forum of Mathematics Pi, **4**(2016), 1–28.

[13] F. J. Chung, *A Partial Data Result for the Magnetic Schrödinger Inverse Problem*, (preprint), arXiv:1111.6658.

[14] F. J. Chung, *A Partial Data Result for the Magnetic Schrödinger Inverse Problem*, Analysis and PDE **7** (2014), 117–157.

[15] D. Dos Santos Ferreira, C. E. Kenig, J. Sjŝtrand and G. Uhlmann, *On the linearized local Calderón problem*, Math. Res. Lett., **16** (2009), 955-970.

[16] D. Dos Santos Ferreira, C. E. Kenig, M. Salo and G. Uhlmann, *Limiting Carleman weights and anisotropic inverse problems*, Invent. Mathematics **178** (2009), 119–171.

[17] D. Dos Santos Ferreira, C. E. Kenig, J. Sjöstrand and G. Uhlmann, *Determining a magnetic Schrödinger operator from partial Cauchy data*, Comm. Mathematical Physics **271** (2007), 467–488.

[18] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics- Second Edition, Vol. 19 (2010).

[19] D. Faraco and K. Rogers, *The Sobolev norm of characteristic functions with applications to the Calderón Inverse Problem*, Quart. J. Math. **64** (2013), 133-147.

[20] D. Holder, *Electrical Impedance Tomography*, Institute of Physics Publishing, Bristol and Philadelphia (2005).

[21] D. Holder, D. Isaacson, J. Müller and S. Siltanen, *Physiological Measurements.* **25**(2003).

[22] B. Haberman and D. Tataru, *Uniqueness in Calderóns problem with Lipschitz conductivities*, Duke Math. J. **162** (2013), 497516.

[23] H. Heck and J. N. Wang, *Stability estimates for the inverse boundary value problem by partial Cauchy data*, Inverse Problems **22** (2006), 1787–1796.

[24] H. Heck and J. N. Wang, *Optimal stability estimate of the inverse boundary value problem by partial measurements*, preprint (2007) arXiv:0708.3289v1.

[25] V. Isakov, *On uniqueness in the inverse conductivity problem with local data*, Inverse Probl. Imaging **1** (2007), 95–105.

[26] C.E. Kenig, J. Sjöstrand and G. Uhlmann, *The Calderón problem with partial data*, Annals Math. **165** (2007), 567–591.

[27] K. Krupchick, M. Lassas and G. Uhlmann, *Inverse problems with partial data for a magnetic Schrödinger operator in an infinite slab and on a bounded domain*, Commun. Math. **313** (2012), 87-126.

[28] K. Krupchyk, G. Uhlmann, *Uniqueness in an inverse boundary problem for a magnetic Schrödinger operator with a bounded magnetic potential*, Comm. Math. Phys., **327** (2014), 993-1009.

[29] R. Kohn and M. Vogelius, *Determining conductivity by boundary measurements*, Comm. Pure Appl. Math., **37** (1984), 289–298.

[30] R. V. Kohn and M. Vogelius, *Determining conductivity by boundary measurements II. Interior results*, Comm. on Pure and Appl. Math., **38** (1985), 643–667.

[31] J. L. Mueller and S. Siltanen, *Linear and Nonlinear Inverse Problems with Practical Applications*, SIAM Computational Science - Engineering, **2012**.

[32] N. Mandache, *Exponential instability in an inverse problem of the Schrödinger equation*, Inverse Problems, **17** (2001), 1435–1444.

[33] F. Natterer, *The Mathematics of Computarized Tomography*, Mc Jhon Wiley, **1986**.

[34] A. Nachman, *Reconstruction from boundary measurements*, Ann. of Math., **128** (1988), 531–576.

[35] A. Nachman and B. Street, *Reconstruction in the Calderón problem with partial data*, Comm. Partial Differential Equations **35** (2010), 375–390.

[36] G. Nakamura, Z. Sun, G. Uhlmann, *Global identifiability for an inverse problem for the Schrödinger equation in a magnetic field*, Math. Ann. **303** (1995), 377–388.

[37] G. Nakamura and G. Uhlmann, *Global uniqueness for an inverse boundary problem arising in elasticity*, Invent. Math. **118** (1994), 457–474.

[38] G. Nakamura and G. Uhlmann, *Erratum. Global uniqueness for an inverse boundary problem arising in elasticity*, Invent. Math. **152** (2003), 205–207.

[39] L. Potenciano-Machado, *Stability estimates for a magnetic Schrödinger operator with partial data*, preprint (2016).

[40] L. Potenciano-Machado, *Optimal stability estimates for a magnetic Schrödinger opeartor with partial data*, preprint (2016).

[41] M. Salo, *Inverse problems for nonsmooth first-order perturbations of the Laplacian*, Ann. Acad. Sci. Fenn. Math. Diss. **139** (2004).

[42] M. Salo, *Semiclassical pseudodifferential calculus and the reconstruction of a magnetic field*, Comm. Partial Differential Equations **31** (2006), 1639–1666.

[43] M. Salo and L. Tzou, *Carleman estimates and inverse problems for Dirac operators*, Math. Ann. **344**(2009), 161–184.

[44] M. Salo, *Inverse problems for nonsmooth first-order perturbations of the Laplacian*, Academi Scientiarum Fennic. Annales Mathematica Dissertationes, **139**, (2004).

[45] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press (1970).

[46] Z. Sun, *An inverse boundary value problem for the Schrödinger operator with vector potentials*, Trans. Amer. Math. Soc. **338** (1992), 953–969.

[47] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math., **125**(1987), 153–169.

[48] C. F. Tolmasky, *Exponentially growing solutions for nonsmooth first-order perturbations of the Laplacian*, SIAM J. Math. Anal. **29**(1998), 116–133.

[49] H. Triebel, *Function spaces in Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers.*, Revista Matemática Complutense **15**, no.2 (2002).

[50] L. Tzou, *Stability estimates for coefficients of the magnetic Schrödinger equation from full and partial boundary measurements*, Comm. Partial Differential Equations **33** (2008), 1911–1952.

[51] G. Uhlmann, *Developments in inverse problems since Calderón's foundational paper*, Chapter 19 - Harmonic Analysis and Partial Differential Equations, University Chicago Press, **1999**.

[52] G. Uhlmann, *Commentary on Calderón's paper* 28*, On an Inverse Boundary Value Problem*, Selected Papers of A.P. Calderón, AMS (2008), 623-636.

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